Lecture 9: Limsup, Liminf; Power Series; Continuous Functions; Exponential Function

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1 Limsup and Liminf

Let a_n be a sequence. From a_n we can form another sequence

$$b_n = \sup\{a_n, a_{n+1}, \ldots\}.$$

Assume there exists a C such that for all n, $a_n \leq C$ (i.e. a_n is bounded from above). Then

$$b_{n+1} = \sup\{a_{n+1}, a_{n+2}, \ldots\} \leq \sup\{a_n, a_{n+1}, \ldots\} = b_n,$$

so b_n is decreasing. If also b_n is bounded from below, then

$$b_n \downarrow b = \inf\{b_i\}.$$

Likewise, if a_n is any sequence bounded from below, set

$$c_n = \inf\{a_n, a_{n+1}, \ldots\}.$$

Then

$$c_{n+1} = \inf\{a_{n+1}, a_{n+2}, \ldots\} \geq \inf\{a_n, a_{n+1}, \ldots\} = c_n,$$

so c_n is increasing. If c_n is bounded from above, then

$$c_n \uparrow c = \sup\{c_i\}.$$

We write

$$b = \limsup_n a_n, \qquad c = \liminf_n a_n.$$

2 Power Series

Last time a_n was a sequence; the power series is $\sum_{n=0}^{\infty} a_n x^n$. For which x does it converge?

Example from last time. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $a_n = \frac{1}{n!}$. For x fixed, $\sum_{n=0}^{\infty} d_n$ with $d_n = \frac{x^n}{n!}$. Using the ratio test,

$$\frac{d_{n+1}}{d_n} = \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1} \to 0.$$

Root Test (adult version)

For $\sum_{n=0}^{\infty} d_n$,

$$\limsup_{n\to\infty} \sqrt[n]{|d_n|} = d,$$

with

$$\begin{cases} d < 1 & \text{convergent,} \\ d > 1 & \text{divergent,} \\ d = 1 & \text{inconclusive.} \end{cases}$$

Radius of Convergence

For $\sum_{n=0}^{\infty} a_n x^n$, set $d_n = a_n x^n$ and apply the root test. For what x does the power series converge?

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

since $\sqrt[n]{|a_n x^n|} = \sqrt[n]{|a_n|}\,|x|$.

The radius of convergence is

$$R = \frac{1}{\limsup_n \sqrt[n]{|a_n|}} = \begin{cases} 0 & \text{if } \limsup_n \sqrt[n]{|a_n|} = \infty, \\ \infty & \text{if } \limsup_n \sqrt[n]{|a_n|} = 0. \end{cases}$$

For $\sum_{n=0}^{\infty} a_n x^n$ with R the radius of convergence: for $|x| < R$ the power series is convergent, for $|x| > R$ the power series is divergent, and at $|x| = R$ (the endpoints $\pm R$) it is unclear – further tests needed.

Proof. By the root test (the limsup version),

$$\begin{aligned} |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 &\iff |x| < R, \\ |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1 &\iff |x| > R, \\ |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1 &\iff |x| = R. \end{aligned}$$

Q.E.D.

Root Test, Version 2.0

For $\sum_{n=0}^{\infty} d_n$,

$$\limsup_{n} \sqrt[n]{|d_n|} = \begin{cases} < 1 & \text{conv,} \\ > 1 & \text{div,} \\ = 1 & \text{inconclusive.} \end{cases}$$

Proof. Suppose $\limsup_{n\to\infty} \sqrt[n]{|d_n|} = d < 1$. Choose $\bar{d}$ with $d < \bar{d} < 1$. Then

$$\sup\{ \sqrt[n]{|d_n|},\ \sqrt[n+1]{|d_{n+1}|}, \ldots\} \leq \bar{d},$$

so there exists N such that for $n \geq N$,

$$\sqrt[n]{|d_n|} \leq \bar{d} \implies |d_n| \leq \bar{d}^n = \bar{d}^n,$$

a geometric series where $\bar{d} < 1$. **Q.E.D.**

3 Exponential Function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \text{radius of convergence} = \infty.$$

Want to show that this is the exponential. We have $E(0) = 1$. Next time we will show

$$E(x+y) = E(x)E(y).$$

If $x \geq 0 \implies E(x) > 0$. If $x < 0 \implies -x > 0$, and

$$1 = E(0) = E(x + (-x)) = E(x)E(-x) \implies E(x) > 0.$$

Defining e^x

Let $e = E(1)$. Want to define a second function, e^x.

For $m \in \mathbb{N}$,

$$e^m = \underbrace{e \cdot e \cdot e \cdots}_{m \text{ times}} = \underbrace{E(1) \cdot E(1) \cdots}_{m \text{ times}} = E(m),$$

and $e^0 = 1 = E(0)$.

Suppose $m < 0$. Then $e^m = \dfrac{1}{e^{-m}}$, since

$$1 = E(0) = E(m + (-m)) = E(m)E(-m) \implies E(m) = \frac{1}{E(-m)} = \frac{1}{e^{-m}} = e^m.$$

Now let $q \in \mathbb{Q}$, $q = \dfrac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Define $e^q = \alpha$ with $\alpha > 0$ by $\alpha^n = e^m$, i.e. $e^q = e^{m/n}$. Need to check that $e^q = E(q)$:

1. $E(q) > 0$. ✓
2. $(E(q))^n = e^m = E(m)$. Need to check $\left(E\left(\frac{m}{n}\right)\right)^n = E(m)$:

$$\underbrace{E\left(\tfrac{m}{n}\right) \cdot E\left(\tfrac{m}{n}\right) \cdots}_{n \text{ times}} = E\left(n \cdot \tfrac{m}{n}\right) = E(m).$$

So $E(q) = e^q$ for all $q \in \mathbb{Q}$.

4 Continuous Functions

Consider e^x, $x \in \mathbb{R}$. This function should be continuous.

Theorem 1: *If f, g are two continuous functions on $\mathbb{R}$ that agree on all of $\mathbb{Q}$, then $f = g$ everywhere. (Will be proved later.)*

Definition. $f : I \to \mathbb{R}$ is said to be continuous at $x_0 \in I$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

f is said to be continuous if it is continuous at all points.

Example. f the constant function, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c$.

Proof of continuity. Given $\varepsilon > 0$, let $\delta = 1$. Then

$$|x - x_0| < 1 \implies |f(x) - f(x_0)| = |c - c| = 0 < \varepsilon.$$

Q.E.D.

Example. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$. This is also continuous.

Proof. Given $\varepsilon > 0$, set $\delta = \varepsilon$. Then

$$|x - x_0| < \varepsilon \implies |f(x) - f(x_0)| = |x - x_0| < \varepsilon.$$

Q.E.D.

5 Algebraic Properties of Continuous Functions

1. If f, g are continuous, then the sum is also continuous.
2. If f is continuous and c is a constant, then $c \cdot f$ is continuous.
3. If f, g are continuous, then $f \cdot g$ is continuous.
4. If f is continuous and $f \neq 0$, then $\dfrac{1}{f}$ is continuous.
5. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, then $h(x) = g(f(x))$ is also continuous.

Example. $f(x) = x^2 + 1$ continuous, $g(x) = x$ continuous, $h(x) = 1$ continuous. Then $x \mapsto g(x)g(x)$ is also continuous, i.e. $g^2(x) + h(x)$ is continuous.

Theorem 2: *Any polynomial is continuous.*

Follows from algebraic rules (1)–(3).

Theorem 3: *All rational functions $\dfrac{P_1(x)}{P_2(x)}$ are continuous.*

Proof. The theorem above, plus rule (4). **Q.E.D.**

6 Product of Continuous Functions

Consider $f(x) = \dfrac{1}{x}$, $x > 0$. We want

$$|f(x)g(x) - f(x_0)g(x_0)| < \varepsilon \quad \text{when} \quad |x - x_0| < \delta.$$

Add and subtract $f(x_0)g(x)$:

$$|f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)| \leq |g(x)(f(x)-f(x_0))|+|f(x_0)(g(x)-g(x_0))|.$$