Lecture 8: Convergence Tests & Power Series

Conor

August 8, 2026

1 Series

Let a_n be a sequence. Form a new sequence S_n by

$$S_n = \sum_{i=1}^{n} a_i.$$

Write $\sum_{i=1}^{\infty} a_i$ for the sequence S_n; it converges if the sequence S_n converges.

Recall that by the Cauchy Convergence Theorem, S_n converges if and only if S_n is a Cauchy sequence:

$$|S_n - S_m| < \varepsilon \quad \text{provided } m, n > N = N(\varepsilon).$$

In particular, taking $m = n - 1$,

$$S_n - S_m = a_1 + \cdots + a_n - (a_1 + \cdots + a_{n-1}) = a_n,$$

so $a_n \to 0$ as $n \to \infty$.

2 Geometric Series

For $c \in \mathbb{R}$,

$$\sum_{i=0}^{\infty} c^i = \frac{1}{1-c},$$

which converges for $|c| < 1$, and is divergent for $|c| = 1$ and $|c| > 1$.

3 Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. It sits inside a larger family:

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \begin{cases} \text{divergent} & \alpha = 1, \\ \text{convergent (integral test)} & \alpha > 1. \end{cases}$$

4 Absolute Convergence

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ converges. It is easy to see that absolute convergence implies convergence.

Alternating harmonic series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \qquad a_n = \frac{(-1)^n}{n}, \qquad |a_n| = \frac{1}{n}.$$

The series of absolute values is the harmonic series, which diverges. The partial sums S_n oscillate: $S_1 = 1$, then S_2, S_3 close in toward the limit from alternating sides.

5 Series with Nonnegative Terms

For $\sum_{n=0}^{\infty} a_n$ with $a_n \geq 0$ for all n, the partial sums S_n are increasing. Hence S_n converges if and only if S_n is bounded from above.

6 Convergence Tests

Three tests: (1) comparison, (2) ratio, (3) root.

Comparison Test, Version 1

Theorem 1 (Comparison Test, Version 1)**:** *Given* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *with* $0 \leq a_n \leq b_n$*: if* $\sum_{n=0}^{\infty} b_n$ *is convergent, then* $\sum_{n=0}^{\infty} a_n$ *is convergent.*

Proof. By the Cauchy Convergence Theorem. Let $S_n^a = a_1 + \cdots + a_n$ and $S_m^b = b_1 + \cdots + b_m$. For $m \leq n$,

$$|S_n^a - S_m^a| = |a_{m+1} + \cdots + a_n| \leq b_{m+1} + \cdots + b_n = S_n^b - S_m^b.$$

Q.E.D.

Example. $\sum_{n=1}^{\infty} \frac{2^{-n}}{n}$. Is this convergent? Set $a_n = \frac{2^{-n}}{n}$ and $b_n = 2^{-n}$. Then $0 \leq a_n \leq b_n$, and $\sum b_n$ is a convergent geometric series with $c = \frac{1}{2}$. Therefore a_n is convergent too.

Comparison Test, Version 2 (Limit Comparison)

Theorem 2 (Comparison Test, Version 2)**:** *Given* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *with* $a_n, b_n \geq 0$ *and* $b_n \neq 0$*, suppose*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L, \qquad L \neq 0,\ L \neq \infty.$$

Then $\sum a_n$ *converges if and only if* $\sum b_n$ *converges.*

Example.

$$a_n = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}, \qquad b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}, \qquad \frac{1}{n^2 - 1} > \frac{1}{n^2}.$$

Then

$$\frac{a_n}{b_n} = \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \frac{n^2}{n^2 - 1} = \frac{1}{1 - \frac{1}{n^2}} \to 1.$$

Ratio Test

Based on comparison between geometric series. For $\sum_{n=0}^{\infty} a_n$, let

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = a.$$

1. $a < 1 \implies$ convergent.
2. $a > 1 \implies$ divergent.
3. $a = 1 \implies$ test is inconclusive.

Proof of Case 1. We need to show that if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = a < 1$ then the series is convergent. Given a with $a < \alpha < 1$, there exists N such that for $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| \leq \alpha, \qquad \text{i.e.} \qquad |a_{n+1}| \leq \alpha |a_n|.$$

Then, starting at $n = N$:

$$|a_{N+1}| \leq \alpha |a_N|, \qquad |a_{N+2}| \leq \alpha |a_{N+1}| \leq \alpha \cdot \alpha |a_N| = \alpha^2 |a_N|,$$

and this continues for all subsequent terms. **Q.E.D.**

Proof of Case 2. $a > 1$. This time choose α with $1 < \alpha < a$. For $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| \geq \alpha \implies |a_{n+1}| \geq \alpha |a_n|.$$

This means the tail does not go to zero, and the series diverges. **Q.E.D.**

Root Test

For $\sum_{n=0}^{\infty} a_n$, consider $\sqrt[n]{|a_n|} = |a_n|^{1/n}$. Suppose

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = c.$$

1. $c < 1 \implies$ convergent.

2. $c > 1 \implies$ divergent.

3. $c = 1 \implies$ inconclusive.

Proof of Cases 1 and 2. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = c < 1$. There exists N such that for $n \geq N$,

$$\sqrt[n]{|a_n|} \leq \gamma \implies |a_n| \leq \gamma^n = b_n.$$

If γ satisfies $c < \gamma < 1$, then $b_n = \gamma^n$ is a convergent geometric series and $\sum a_n$ converges. If instead $c > 1$ and γ satisfies $1 < \gamma < c$, then γ^n is a divergent geometric series and $\sum a_n$ diverges. **Q.E.D.**

7 Power Series

$\sum_{n=0}^{\infty} x^n$ is convergent if $|x| < 1$. In general, if a_n is a sequence, then

$$\sum_{n=0}^{\infty} a_n x^n$$

is a *power series.* Examples:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The central question: for which x does $\sum_{n=0}^{\infty} a_n x^n$ converge?

Example. $\sum_{n=0}^{\infty} x^n$. For each fixed x this is the geometric series: it converges for $|x| < 1$ and diverges for $|x| \geq 1$.

Example. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. For what x does this converge? For $x \neq 0$, let $b_n = \frac{x^n}{n!}$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \left|\frac{x^{n+1}\, n!}{x^n\,(n+1)!}\right| = \left|\frac{x}{n+1}\right| \to 0.$$

By the ratio test, this is convergent (for all x).