Lecture 7: Bolzano–Weierstrass Theorem, Series

Conor

August 6, 2026

1 Cauchy Convergence Theorem

Any Cauchy sequence is convergent. The proof uses the Bolzano–Weierstrass theorem, i.e. that any bounded sequence has a convergent subsequence.

2 Proof of the Bolzano–Weierstrass Theorem

Theorem 1 (Bolzano–Weierstrass)**:** *Any bounded sequence has a convergent subsequence.*

Proof. Take $a_n \in [0, 1]$. We want to find a subsequence a_{n_k} and two other sequences b_k, c_k where

$$b_k \leq a_{n_k} \leq c_k,$$

so that a_{n_k} is squeezed between the two others. We arrange b_k increasing and c_k decreasing, with $b_k \leq c_1$ and $b_1 \leq c_k$. We can then use the squeeze theorem to prove that a_{n_k} is convergent: if $b_k \to b$ and $c_k \to c$ with $b = c$, then a_{n_k} converges to that common value.

Construct the sequences by repeated bisection. Start with $b_1 = 0$, $c_1 = 1$, and $a_{n_1} = a_1$, so

$$0 = b_1 \leq a_{n_1} \leq c_1 = 1.$$

Assume infinitely many a_n lie in $[0, \frac{1}{2}]$. Let a_{n_2} be the second element in the original sequence that lies in $[0, \frac{1}{2}]$, and set $b_2 = 0$, $c_2 = \frac{1}{2}$.

Now assume infinitely many a_n lie in $[\frac{1}{4}, \frac{1}{2}]$. Let a_{n_3} be the third element in the original sequence that lies in $[\frac{1}{4}, \frac{1}{2}]$, and set $b_3 = \frac{1}{4}$, $c_3 = \frac{1}{2}$.

By induction, a_{n_k} converges. **Q.E.D.**

3 Series

Let a_n be a sequence. Form another sequence S_n by

$$S_1 = a_1, \qquad S_2 = a_1 + a_2, \qquad S_3 = a_1 + a_2 + a_3, \qquad S_n = \sum_{i=1}^{n} a_i.$$

S_n is said to be a *series*.

4 Geometric Series

Consider

$$\sum_{i=0}^{n} c^i.$$

This is very useful for comparing to other series. Note that

$$\sum_{i=0}^{n} c^i = \sum_{i=1}^{n} c^i,$$

and to denote the limit of the series we use $\sum_{i=0}^{\infty} a_i$.

Since

$$(1-c)\sum_{i=0}^{n} c^i = 1 - c^{n+1},$$

assuming $c \neq 1$,

$$\boxed{\sum_{i=0}^{n} c^i = \frac{1-c^{n+1}}{1-c}}$$

Theorem 2 (Geometric Series Convergence)**:** *The geometric series* $\displaystyle\sum_{i=0}^{\infty} c^i$ *is convergent if* $|c| < 1$ *and divergent if* $|c| \geq 1$. *Moreover, if* $|c| < 1$ *the limit is*

$$\frac{1}{1-c} = \sum_{i=0}^{\infty} c^i.$$

Proof. Assume $|c| < 1$. Then

$$S_n = \sum_{i=0}^{n} c^i = \frac{1-c^{n+1}}{1-c} \to \frac{1}{1-c},$$

since $|c^{n+1}| = |c|^{n+1} \to 0$.

Assume instead $|c| \geq 1$. Then

$$|S_{n+1} - S_n| = |c^{n+1}| = |c|^{n+1} \geq 1.$$

This is no longer a Cauchy sequence, therefore it does not converge. **Q.E.D.**

5 Harmonic Series

Theorem 3: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ *does not converge.*

Proof.

Claim 4: $S_{2^n-1} = \displaystyle\sum_{i=1}^{2^n-1} \frac{1}{i} \geq \frac{n}{2}$.

For $n = 1$: we want to show $S_1 \geq \frac{1}{2}$. Since $S_1 = 1$, the claim is okay for $n = 1$.

Assume the claim is correct for n; show it is correct for $n+1$. We want to show $S_2 \geq \frac{2}{2}$. Since $S_2 = \frac{3}{2}$, the claim is okay for $n = 2$.

By induction the claim is always true. Since n can be arbitrarily large, the theorem is proven true. **Q.E.D.**

6 Necessary Condition for Convergence

For a series $\displaystyle\sum_{n=0}^{\infty} a_n$, always check whether $a_n \to 0$. Since $S_n - S_{n-1} = a_n$, if a_n does not tend to 0 then the series does not converge.

If $a_n \geq 0$, then S_n is increasing: for $\displaystyle\sum_{n=0}^{\infty} a_n$,

$$S_{n+1} = \sum_{i=0}^{n} a_i + a_{n+1} = S_n + a_{n+1} \geq S_n.$$

7 Absolute Convergence

$\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely if $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem 5: *Absolute convergence implies convergence.*

Proof. Let $S_n = \displaystyle\sum_{i=0}^{n} a_i$ and $\overline{S}_n = \displaystyle\sum_{i=0}^{n} |a_i|$. For $n \geq m$,

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = \overline{S}_n - \overline{S}_m.$$

$\overline{S}_n$ is a Cauchy sequence because it converges. Because $|S_n - S_m| \leq |\overline{S}_n - \overline{S}_m|$, S_n must converge too. **Q.E.D.**

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, called the alternating harmonic series, is convergent but not absolutely convergent.

8 A Convergent Series: $\sum 1/n^2$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Here $a_n = \frac{1}{n^2} \geq 0$, so we need to show the partial sums $S_n = \sum_{i=1}^{n} \frac{1}{i^2}$ are bounded.

Claim 6: $S_{2^n-1} \leq \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i \leq \frac{1}{1-\frac{1}{2}} = 2,$

where the middle expression is a geometric series with $c = \frac{1}{2}$, hence convergent.

Proof. For $n = 1$: $S_1 \leq 1$. We need to show this; since $S_1 = 1$, okay.

Assume $S_{2^n-1} \leq \sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i$; we want to show it for $n+1$. Then

$$\begin{aligned} S_{2^{n+1}-1} &= S_{2^n-1} + \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i^2} \\ &\leq S_{2^n-1} + 2^n \left(\frac{1}{2^n}\right)^2 \\ &= \sum_{i=0}^{n-1} \left(\tfrac{1}{2}\right)^i + \frac{1}{2^n} = \sum_{i=0}^{n} \left(\tfrac{1}{2}\right)^i. \qquad \text{OK} \end{aligned}$$

By induction, converges. **Q.E.D.**

9 Convergence Tests

To determine if a series converges there are a number of tests: comparison test, ratio test, root test, integral test, alternating series test.

Comparison test. Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$: if $0 \leq a_n \leq b_n$ and $b_n \to b$, then $a_n \to a$.