Lecture 6 Notes

Conor

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1 Review

1.1 Monotone Convergence Theorem

(a) $x_n \leq x_{n+1} \leq \cdots \leq M$ (increasing)

(b) $x_n \geq x_{n+1} \geq \cdots \geq m$ (decreasing)

Then $x_n \to \sup\{x_n\}$ in case (a), and $x_n \to \inf\{x_n\}$ in case (b).

1.2 Cauchy Sequence

A sequence x_n is said to be a Cauchy sequence if for $\varepsilon > 0$, $\exists\ N$ such that $m, n \geq N \Rightarrow |x_n - x_m| < \varepsilon$. In plain speak, the tail of the sequence gets close together.

Theorem 1: *If x_n is a Cauchy sequence, then x_n is bounded.*

Proof. Take $\varepsilon = 1$ in the definition of a Cauchy sequence. Then $\exists\ N$ such that if $n, m \geq N$, $|x_n - x_m| < 1$. In particular, setting $m = N$, for all $n \geq N$,

$$|x_n - x_N| < 1.$$

For $n \geq N$,

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < 1 + |x_N|.$$

Then

$$\max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\} = M$$

bounds the sequence: $\forall\ n$, $|x_n| \leq M$. **Q.E.D.**

2 Cauchy Convergence Theorem

Theorem 2 (Cauchy Convergence Theorem)**:** *Any Cauchy sequence is convergent.*

Recall that last time we showed the converse: any convergent sequence is a Cauchy sequence.

3 Applications

3.1 The Contraction Mapping Theorem

A map $T : \mathbb{R} \to \mathbb{R}$ is said to be a contraction map if $\exists\ 1 > C > 0$ such that

$$|T(x) - T(y)| \leq C|x - y|.$$

Recall what a fixed point is.

Lemma 3: *If $T : \mathbb{R} \to \mathbb{R}$ is a contraction map, then there cannot exist more than one fixed point.*

Proof. Suppose x, y are fixed points. Then, with $0 < C < 1$,

$$C|x - y| \geq |T(x) - T(y)| = |x - y|.$$

This forces $|x - y| = 0$, so $x = y$. **Q.E.D.**

Now let T be a contraction map and x any point in the domain. Define

$$a_0 = x, \quad a_1 = T(x) = T(a_0), \quad a_2 = T(a_1) = T(T(x)) = T^2(x),$$
$$a_3 = T(a_2) = T(T(a_1)) = T(T(T(x))) = T^3(x), \quad \ldots, \quad a_{n+1} = T(a_n) = T^{n+1}(x).$$

3.2 a_n is a Cauchy sequence

The key starting point is that

$$|a_{n+1} - a_n| = |T(a_n) - T(a_{n-1})| \leq C|a_n - a_{n-1}|.$$

We can then use induction to prove that as $n \to \infty$, the gaps $|a_{n+1} - a_n| \to 0$, from which a_n is Cauchy.

3.3 Proof of the Contraction Mapping Theorem

Theorem 4 (Contraction Mapping Theorem)**:** *A contraction map has a fixed point.*

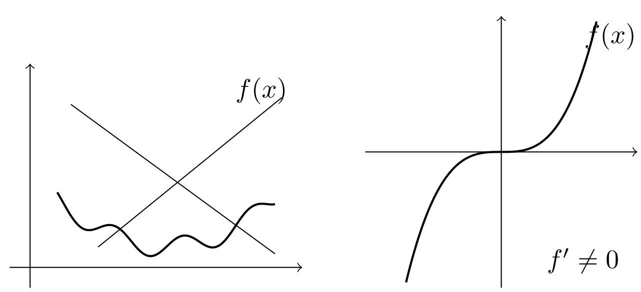
Proof. Let $x \in \mathbb{R}$ and set $a_n = T^n(x)$. We have that a_n is a Cauchy sequence, so by the Cauchy Convergence Theorem, $a_n \to a$. We want to show a is a fixed point for T.

Since $a_n \to a$, also $a_{n+1} \to a$. Note $T(a_n) = a_{n+1}$. We want to show $T(a) = a$. Now

$$|T(a_n) - T(a)| \leq C|a_n - a|.$$

Since $a_n \to a$, $|a_n - a| \to 0$, so $|a_{n+1} - T(a)| \to 0$, i.e. $a_{n+1} \to T(a)$. But we also have $a_{n+1} \to a$. Therefore $T(a) = a$. **Q.E.D.**

3.4 Newton's Method



The idea: define

$$T(x) = x - \frac{f(x)}{f'(x)}.$$

Then

$$T'(x) = 1 - \frac{(f')^2 - ff''}{(f')^2} = 1 - \frac{(f')^2}{(f')^2} + f \cdot \frac{f''}{(f')^2} = f(x) \cdot \frac{f''(x)}{(f'(x))^2}.$$

By the Mean Value Theorem, $T(x) - T(y) = T'(\xi)(x - y)$ for some ξ between x and y. If we knew $|T'| \leq C < 1$, then

$$|T(x) - T(y)| = |T'(\xi)|\,|x - y| \leq C|x - y|,$$

so T is a contraction. Start with a "good" guess: where $|f(x)|$ is very small. This means $T'(x)$ would be very small, since by definition $T'(x) = f(x) \cdot \frac{f''(x)}{(f'(x))^2}$.

Iterating $x, T(x), T^2(x), \ldots$ gives $T^n(x) \to x_0$, the fixed point for T. At the fixed point, with $T(x) = x - \frac{f(x)}{f'(x)}$,

$$T(x_0) = x_0 = x_0 - \frac{f(x_0)}{f'(x_0)} \implies -\frac{f(x_0)}{f'(x_0)} = 0 \implies f(x_0) = 0$$

(since $f'(x_0) \neq 0$). So the fixed point of T is a root of f.

4 The Bolzano–Weierstrass Theorem

Theorem 5 (Bolzano–Weierstrass)**:** *Every bounded sequence has a convergent subsequence.*

(Proven next time.) From the Bolzano–Weierstrass theorem we get the Cauchy Convergence Theorem as follows.

Proof of the Cauchy Convergence Theorem. Suppose x_n is a Cauchy sequence. We already showed x_n is bounded, so by Bolzano–Weierstrass there is a convergent subsequence $x_{n_k} \to x$.

Since x_n is Cauchy, given $\varepsilon > 0$, $\exists\ N_1$ such that $m, n \geq N_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2}$. Since $x_{n_k} \to x$, $\exists\ N_2$ such that $k \geq N_2 \Rightarrow |x_{n_k} - x| < \frac{\varepsilon}{2}$.

Set $N = \max\{N_1, n_{N_2}\}$. For $n \geq N$,

$$|x - x_n| = |x - x_N + x_N - x_n| \leq |x - x_N| + |x_N - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

5 Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at x_0 if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

A function is continuous if it is continuous at all points.

5.1 An Application of the Cauchy Convergence Theorem

Theorem 6: *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x_n \to x$, then $f(x_n) \to f(x)$.*

Proof. Since f is continuous at x, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Since $x_n \to x$, $\exists\ N$ such that $n \geq N \Rightarrow |x_n - x| < \delta$. Therefore for $n \geq N$,

$$|f(x_n) - f(x)| < \varepsilon.$$

Q.E.D.

6 The Extreme Value Theorem

Theorem 7 (Extreme Value Theorem)**:** *If $f : [a, b] \to \mathbb{R}$ is continuous, then $\sup_{[a,b]}\{f\}$ is achieved at some $x_M \in [a, b]$:*

$$\sup_{[a,b]}\{f\} = f(x_M).$$

Moreover, $\inf_{[a,b]}\{f\} = f(x_m)$ for some $x_m \in [a, b]$.

Proof (focusing on sup). There exists a sequence $x_n \in [a, b]$ with $f(x_n) \to \sup f$. Note x_n is a bounded sequence, so by Bolzano–Weierstrass there is a convergent subsequence $x_{n_k} \to x_M \in [a, b]$. By continuity, $f(x_{n_k}) \to f(x_M)$. But $f(x_{n_k}) \to \sup f$, so

$$\sup f = f(x_M).$$

Q.E.D.