Lecture 5 Notes

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1 An Opening Example

Consider the sequence

$$a_n = \frac{n^2+1}{n^2+n+1} = \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{1+\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^2}}.$$

Define the auxiliary sequences $x_n = \frac{1}{n} \to 0$, $y_n = \frac{1}{n^2} \to 0$, and $z_n = 1 \to 1$. Then the numerator is

$$b_n = z_n + y_n \to 1,$$

and the denominator is

$$c_n = z_n + x_n + y_n \to 1 = c.$$

Since $c_n \neq 0$ and $c \neq 0$, the reciprocal rule gives $\frac{1}{c_n} \to \frac{1}{c} = 1$, and the product rule gives

$$b_n \cdot \frac{1}{c_n} = a_n \to 1 \cdot 1 = 1.$$

2 Monotone Sequences

A sequence a_n is *monotone increasing* if $\forall\ n \in \mathbb{N}$, $a_n \leq a_{n+1}$. It is *monotone decreasing* if $\forall\ n \in \mathbb{N}$, $a_n \geq a_{n+1}$.

Theorem 1 (Monotone Convergence Theorem)**:** *Suppose that a_n is a monotone increasing sequence that is bounded, i.e. $\exists\ A \in \mathbb{R}$ such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq A$. Then it is convergent, with limit $a = \sup\{a_n \mid n \in \mathbb{N}\}$.*

A parallel theorem exists for monotone decreasing sequences: if a_n is a monotone decreasing sequence that is bounded from below, then $a_n \to a$, where $a = \inf\{a_n \mid n \in \mathbb{N}\}$.

3 Sup and Inf Sequences

Let a_n be any bounded sequence. We can form two other sequences,

$$b_n = \sup\{a_n, a_{n+1}, \ldots\}, \qquad c_n = \inf\{a_n, a_{n+1}, \ldots\}.$$

For example,

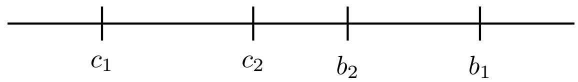
$$b_2 = \sup\{a_2, a_3, \ldots\}, \qquad b_3 = \sup\{a_3, a_4, a_5, \ldots\},$$

and $b_2 \geq b_3$ due to the inclusion of a_2. In general $b_n \geq b_{n+1}$, so b_n is decreasing; $b_n \searrow$ is the notation for a decreasing sequence.

Similarly, with

$$c_n = \inf\{a_n, a_{n+1}, \ldots\}$$

we have $c_{n+1} \geq c_n$, so c_n is increasing; $c_n \nearrow$ is also notation. Note that $c_n \leq b_n$.



For a bounded sequence a_n, the sequences b_n and c_n are both monotone, so

$$b_n \searrow b, \qquad c_n \nearrow c.$$

Furthermore $c \leq b$. This will be proved later, with Cauchy sequences.

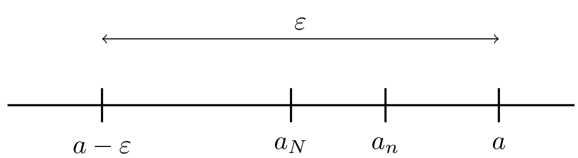
4 Proof of the Monotone Convergence Theorem

Proof (monotone increasing case). By assumption $a_n \leq a_{n+1} \leq A$. Set $a = \sup\{a_n \mid n \in \mathbb{N}\}$. We want to show $a_n \to a$.

Since a is the supremum, it is an upper bound, so $a_n \leq a$ for all n. Given $\varepsilon > 0$, since a is the *least* upper bound, $a - \varepsilon$ is not an upper bound, so $\exists\, N$ with $a - \varepsilon < a_N$. Because the sequence is increasing, for all $n \geq N$,

$$a - \varepsilon < a_N \leq a_n \leq a,$$

which gives $a - a_n < \varepsilon$, i.e. $|a - a_n| < \varepsilon$. **Q.E.D.**



5 Constructing $\sqrt{2}$ by a Monotone Sequence

Proof. We will define two sequences a_n and b_n, where

$$a_n = \frac{b_n}{10^{\,n-1}}, \qquad (a_1 = b_1).$$

The sequence a_n is monotone increasing and $b_n \in \mathbb{N}$, where b_n is the largest natural number such that $b_n^2 \leq 2 \cdot 10^{\,2n-2}$.

For $n = 1$, $b_1 = 1 = a_1$.

5.1 a_n is monotone increasing

We want to show $a_n \leq a_{n+1}$. This is equivalent to showing

$$a_n = \frac{b_n}{10^{\,n-1}} \leq a_{n+1} = \frac{b_{n+1}}{10^{(n-1)+1}} = \frac{b_{n+1}}{10^{\,n}}.$$

By the transitive property it suffices to show $10\, b_n \leq b_{n+1}$:

$$\frac{b_n}{10^{\,n-1}} \leq \frac{b_{n+1}}{10^{\,n}} \implies \frac{10\, b_n}{10^{\,n}} \leq \frac{b_{n+1}}{10^{\,n}} \implies 10\, b_n \leq b_{n+1}.$$

5.2 a_n is bounded

Since a_n is monotone increasing and bounded, a_n is convergent by the Monotone Convergence Theorem:

$$a_n^2 = a_n \cdot a_n = \frac{b_n}{10^{\,n-1}} \cdot \frac{b_n}{10^{\,n-1}} = \frac{b_n^2}{10^{\,2n-2}} \leq 2.$$

5.3 The limit satisfies $a^2 = 2$

Since $a_n \to a$, the product rule gives $a_n \cdot a_n \to a^2$, and since $a_n^2 \leq 2$ for all n, we get $a^2 \leq 2$. It remains to show $a^2 \geq 2$.

By the defining property of b_n as the *largest* natural number with $b_n^2 \leq 2 \cdot 10^{2n-2}$, the next integer overshoots:

$$(b_n + 1)^2 > 2 \cdot 10^{2n-2} \implies \frac{(b_n+1)^2}{10^{2n-2}} > 2.$$

Using $10^{2n-2} = 10^{n-1} \cdot 10^{n-1}$,

$$\left(\frac{b_n + 1}{10^{n-1}}\right)\left(\frac{b_n + 1}{10^{n-1}}\right) = \left(\frac{b_n}{10^{n-1}} + \frac{1}{10^{n-1}}\right)\left(\frac{b_n}{10^{n-1}} + \frac{1}{10^{n-1}}\right) > 2.$$

Each factor is $a_n + \frac{1}{10^{n-1}}$. As $n \to \infty$, $a_n \to a$ and $\frac{1}{10^{n-1}} \to 0$, so both factors tend to a, giving

$$a^2 \geq 2.$$

Combining $a^2 \leq 2$ and $a^2 \geq 2$ yields $a^2 = 2$. **Q.E.D.**

6 Cauchy Sequences

A sequence is a *Cauchy sequence* if it has the property that $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$ such that $\forall\, n, m \geq N$, $|a_n - a_m| < \varepsilon$.

Theorem 2: *A sequence is a Cauchy sequence if and only if it is convergent.*

7 The Contraction Mapping Theorem

A map $T : \mathbb{R} \to \mathbb{R}$ is a *contraction map* if there exists $0 < C < 1$ such that for all $x, y \in \mathbb{R}$,

$$|T(x) - T(y)| \leq C|x - y|.$$

Theorem 3 (Contraction Mapping Theorem)**:** *If $T : \mathbb{R} \to \mathbb{R}$ is a contraction map, then T has a fixed point, i.e. a point where $T(x) = x$.*