Lecture 4 Notes

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1 Definitions

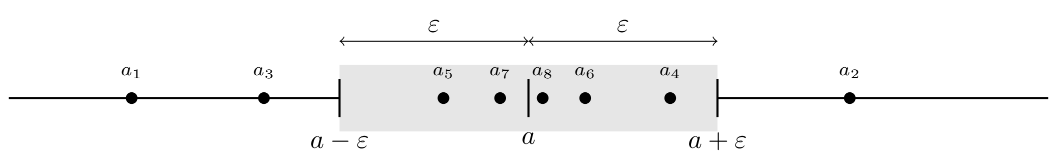
This lecture is all about sequences. A sequence is a function which maps the natural numbers into the real numbers,

$$f : \mathbb{N} \to \mathbb{R}, \qquad a_n = f(n).$$

It is denoted by a letter with a subscript, like a_n. An example of a sequence is $a_1 = 1, a_2 = 1.4, a_3 = 1.41, a_4 = 1.414$. This sequence is approaching $\sqrt{2}$. It is said that $\sqrt{2}$ is the limit of the sequence. However not all sequences need to have a limit. The sequence $a_n = (-1)^n$ does not have a limit; it alternates between -1 and 1 forever. By contrast, the sequence $a_n = \frac{1}{n}$, whose terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, does have a limit, namely 0. The formal definition of a converging sequence is:

A sequence a_n is convergent to $a \in \mathbb{R}$ if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$, $|a_n - a| < \varepsilon$.

In plain english, this means that a sequence is converging if, when you give me some margin for error (ε), I can give you an N that means the sequence never moves outside of those bounds. Equivalently, every term past a_N lands in the open interval $(a - \varepsilon, a + \varepsilon)$. The number line below helps visually illustrate this.



It can be easily seen that for whatever epsilon that was chosen above, the sequence converges to a value a. Using the definition of a converging sequence, we can say that $N = 4$. We can also say that $\lim\limits_{n\to\infty} a_n = a$. If a sequence doesn't converge, we say it diverges.

2 Explorations

We will now use these definitions to construct a simple proof that $.\bar{9} = 1$.

Claim 1: $.\bar{9} = 1$

Proof. Let a_n be the sequence $a_1 = .9,\ a_2 = .99,\ a_3 = .999,\ \ldots$, so that a_n is the decimal with n nines and $a_n = 1 - \frac{1}{10^n}$. Given $\varepsilon > 0$, we need to find N such that $\forall\ n \geq N$, $|a_n - 1| < \varepsilon$. To do that, choose N such that $\frac{1}{N} < \varepsilon$. This can be done using the archimedian property. Note also that $10^n \geq n$ for all $n \geq 1$, so $\frac{1}{10^n} \leq \frac{1}{n}$. Then for any $n \geq N$,

$$|a_n - 1| = \frac{1}{10^n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Q.E.D.

3 Boundedness

Theorem 2: *If a_n is a convergent sequence, then a_n is bounded. That is, the set $\{a_1, a_2, a_3, \ldots\}$ is bounded.*

Note that the converse is not true. A sequence can be bounded without converging, as $a_n = (-1)^n$ demonstrates.

Proof. Since a_n is convergent, $a_n \to a$ for some $a \in \mathbb{R}$. In particular, if we choose $\varepsilon = 1$, then $\exists\, N$ such that $\forall\, n \geq N$, $|a_n - a| < 1$.

The set $\{a_1, \ldots, a_{N-1}\}$ is finite, so $\exists\, C \in \mathbb{R}$ such that $|a_n| \leq C$ for any a_n in this finite set; that is, $|a_n| \leq C$ if $n = 1, \ldots, N-1$.

Let $S = \{|a_n| \mid n = 1, 2, \ldots\}$.

Claim 3: $\sup S \leq \max\{C, |a| + 1\}$, *where the* 1 *came from our choice of* ε.

The finite set is already handled by C, so we only need to show this for $n \geq N$:

$$|a_n| = |(a_n - a) + a| \leq |a_n - a| + |a| \leq 1 + |a|.$$

By the transitive property, this means that $|a_n| \leq 1 + |a|$. This demonstrates it is bounded by the converging value. **Q.E.D.**

4 Algebraic Properties of Limits

Let a_n and b_n be sequences with $a_n \to a$ and $b_n \to b$. Then:

1. For $c \in \mathbb{R}$, if $c_n = c\,a_n$, then $c_n \to ca$.
2. If $c_n = a_n + b_n$, then $c_n \to a + b$.
3. If $c_n = a_n b_n$, then $c_n \to ab$.
4. If $a_n \neq 0$ for all n and $a \neq 0$, and $c_n = \frac{1}{a_n}$, then $c_n \to \frac{1}{a}$.

4.1 Proving Property 1

Proof. Given $\varepsilon > 0$. Assume the non-trivial case ($c \neq 0$). Since $a_n \to a$, $\exists\, N$ such that $\forall\, n \geq N$,

$$|a_n - a| < \frac{\varepsilon}{|c|}.$$

Multiplying through by $|c|$ gives $|c(a_n - a)| < \varepsilon$. **Q.E.D.**

4.2 Proving Property 2

Proof. Given $\varepsilon > 0$. Since $a_n \to a$, $\exists\, N_a$ such that $n \geq N_a \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$. Likewise, since $b_n \to b$, $\exists\, N_b$ such that $n \geq N_b \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$.

Set $N = \max\{N_a, N_b\}$. For $n \geq N$,

$$|c_n - (a+b)| = |a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|.$$

Since $n \geq N \geq N_a, N_b$,

$$|c_n - (a+b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

4.3 Proof of Property 3 (Outline)

Proof. Let $c_n = a_n b_n$. Adding and subtracting $a_n b$,

$$|c_n - ab| = |a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| = |a_n(b_n - b) + (a_n - a)b|$$
$$\leq |a_n(b_n - b)| + |(a_n - a)b| = |a_n||b_n - b| + |a_n - a||b|.$$

Since $\{a_n \mid n = 1, \ldots\}$ is convergent it is bounded, so $\exists\, C$ such that $|a_n| \leq C$. Therefore

$$|c_n - ab| \leq C|b_n - b| + |b||a_n - a|.$$

To prove 3, given $\varepsilon > 0$: since $a_n \to a$, find N_a such that $n \geq N_a \Rightarrow$

$$|a_n - a| < \frac{\varepsilon}{2(|b| + 1)},$$

where the $+1$ appears since $|b|$ could be 0. Likewise, $\exists\, N_b$ such that if $n \geq N_b$,

$$|b_n - b| < \frac{\varepsilon}{2(1 + C)}.$$

Therefore for $n \geq \max\{N_a, N_b\}$,

$$|c_n - ab| < C \cdot \frac{\varepsilon}{2(1 + C)} + |b| \cdot \frac{\varepsilon}{2(|b| + 1)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

4.4 Property 4 Proof

Proof. Let $c_n = \frac{1}{a_n}$ with $a_n \neq 0$ and $a \neq 0$. Then

$$\left| c_n - \frac{1}{a} \right| = \left| \frac{1}{a_n} - \frac{1}{a} \right| = \left| \frac{a - a_n}{a_n a} \right|.$$

We need to bound $|a_n a|$ from below, which is equivalent to bounding $|a_n|$ from below. Writing $a = a - a_n + a_n$ and applying the triangle inequality,

$$|a| \leq |a - a_n| + |a_n| \quad \Longrightarrow \quad |a| - |a - a_n| \leq |a_n|.$$

Since $a \neq 0$, we have $\frac{|a|}{2} > 0$, so $\exists\, N_1$ such that $n \geq N_1 \Rightarrow |a - a_n| < \frac{|a|}{2}$. Combining,

$$\frac{|a|}{2} < |a| - |a - a_n| \leq |a_n|.$$

Then when $n \geq N_1$,

$$\left| c_n - \frac{1}{a} \right| = \frac{|a - a_n|}{|a_n||a|} \leq \frac{2|a - a_n|}{|a|^2}.$$

Therefore when we choose an $N_2 > N_1$, we can force $\frac{2|a - a_n|}{|a|^2}$ to be smaller than any given ε. **Q.E.D.**

5 Subsequences

Consider $a_n = (-1)^n$, so $a_1 = -1$, $a_2 = 1$, and so on. Some subsequences of it are:

1. $b_k = 1$ for all k
2. $b_k = -1$ for all k
3. $b_1 = -1,\ b_2 = -1,\ b_3 = 1,\ b_4 = 1,\ \ldots$

Now consider $a_n = n$. Then $1, 3, 5, 7$ is a subsequence, and $2, 4, 6, 8$ is another subsequence. However, $1, 1, 2, 3, 4$ is *not* a subsequence, since a term is repeated, and $3, 2, 4, 5$ is *not* a subsequence, since it is out of order.

Formal Definition

A subsequence of a_n is $b_k = f(g(k)) = a_{n_k}$, where $g : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

Theorem 4: *a_n is convergent to a if and only if all subsequences of a_n converge to a.*

Proof. One of the two implications is trivial, namely the reverse direction, since a_n is itself a subsequence of a_n.

For the forward direction, given $\varepsilon > 0$, $\exists\ N$ such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$. Take $k \geq N$. Since g is strictly increasing, $n_k = g(k) \geq k \geq N$, and therefore

$$|a_{n_k} - a| < \varepsilon.$$

Q.E.D.

All sequences, including subsequences, are infinite. That is all for lecture 4.