Lecture 23: Existence & Uniqueness for ODEs; the Picard–Lindelöf Theorem

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1 The Picard–Lindelöf Theorem

An ODE is an equation in an unknown function and its derivative:

$$\begin{cases} y' = f(y) + g(x), \\ y(0) = a, \end{cases} \qquad y = y(x), \quad x \in \mathbb{R},$$

where $f, g : \mathbb{R} \to \mathbb{R}$, f is differentiable and f' is continuous, and g is continuous.

Theorem 1 (Picard–Lindelöf)**:** *Given f, g, and a as above, there exists $\delta > 0$ and a solution $y : [-\delta, \delta] \to \mathbb{R}$ to the ODE. Moreover, solutions are unique.*

The proof uses the theory of metric spaces. Let $X = C([a,b])$, and for $h_1, h_2 \in C([a,b])$ set

$$d(h_1, h_2) = \max_{x \in [a,b]} |h_1(x) - h_2(x)|.$$

Theorem 2: *$\big(C([a,b]), d\big)$ is Cauchy complete.*

Another _____ will be a contracting map.

2 Contracting Maps (Generalized)

Definition 3: Let (X, d) be a metric space and $T : X \to X$. Then T is said to be *contracting* if there exists $c < 1$ such that

$$d\big(T(x), T(y)\big) \leq c\, d(x - y).$$

Definition 4: For $T : X \to X$, a *fixed point* is a point $x \in X$ with $T(x) = x$.

If T is contracting, T can have at most one fixed point. If x, y are both fixed points, so $T(x) = x$ and $T(y) = y$, then

$$d(x, y) = d\big(T(x), T(y)\big) = |x - y| \leq c\, d(x, y),$$

hence $d(x, y) = 0$.

Theorem 5: *If (X, d) is a Cauchy complete metric space and $T : X \to X$ is a contracting map, then T has a unique fixed point.*

Proof. Let $x \in X$ and consider

$$x,\ T(x),\ T^2(x),\ T^3(x),\ \dots,\ T^n(x).$$

Claim 6: *$T^n(x) \to x_\infty$, and x_∞ is the fixed point.*

First show T is Cauchy complete. Consider $d\big(T^n(x), T^m(x)\big)$ with $m = n + k$, $k > 0$. Then

$$d\big(T^\ell(x), T^{\ell+1}(x)\big) = d\Big(T\big(T^{\ell-1}(x)\big), T\big(T^\ell(x)\big)\Big) \le c\, d\big(T^{\ell-1}(x), T^\ell(x)\big).$$

Iterating this to apply to the n and m case, it can be seen that given $\varepsilon > 0$, $d\big(T^n(x), T^m(x)\big)$ must be less than epsilon due to the $(c)^n$ term.

Since (X, d) is Cauchy complete and $T^n(x)$ is a Cauchy sequence, it is convergent:

$$T_n(y) \to x_\infty.$$

Claim 7: *x_∞ is a fixed point.*

$$T\big(T^n(x)\big) = T^{n+1}(x) \to x_\infty,$$

but $T^n(x) \to x_\infty$, and therefore $T(x_\infty) = x_\infty$. **Q.E.D.**

3 Construction of the Operator

Let $\big(C([-1,1]), d\big) = (X, d)$ and $y \in C([-1,1])$. Define

$$T : C([-1,1]) \to C([-1,1]), \qquad T(y)(x) = a + \int_0^x \big[f(y(s)) + g(s)\big]\, ds.$$

By the FTC,

$$\big(T(y)\big)' = f(y) + g(x).$$

Note: $T(y)(0) = a$.

If y is a fixed point, then $T(y) = y$, so

$$y' = f(y) + g(x), \qquad a = T(y)(0) = y(0).$$

Need to show that T is a contracting map, $T : \overline{B}_R(0) \to \overline{B}_R(0)$, where

$$\overline{B}_R(0) = \{\, y \in C([-1,1]) \mid |y| \le R \,\} = \{\, y \in C([-1,1]) \mid d(y, 0) \le R \,\}.$$

Set $R = |a| + 2$, and

$$L_1 = \max_{|z| \le R} |f(z)|, \qquad L_2 = \max_{|x| \le 1} |g(x)|.$$

First Step

There exists $\delta_0 > 0$ with $y \in C([-\delta_0, \delta_0])$ and $|y| \le R \iff y \in \overline{B}_R(0)$. We want to show $|T(y)| \le R$, i.e. that $y \in \overline{B}_R(0) \implies T(y) \in \overline{B}_R(0)$.

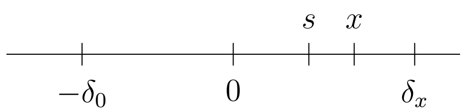
Take

$$\delta_0 = \min\left\{1,\ \frac{1}{L_1+1},\ \frac{1}{L_2+1}\right\}, \qquad [-\delta_0, \delta_0] \subseteq [-1, 1].$$

Then

$$\begin{aligned}
|T(y)(x)| &= \left| a + \int_0^x \big[f(y(s)) + g(s)\big]\, ds \right| \\
&\le |a| + \left| \int_0^x \big[f(y(s)) + g(s)\big]\, ds \right| \\
&\le |a| + \int_0^x \big[|f(y(s))| + |g(s)|\big]\, ds.
\end{aligned}$$

Here s is between 0 and x:



Since $s \in [-\delta_0, \delta_0] \subseteq [-1, 1] \implies |g(s)| \le L_2$, and $|y| \le R \implies f(y(s)) \le L_1$,

$$\begin{aligned}
T(y)(x) &\le |a| + \int_0^x \big[|f(y(s))| + |g(s)|\big]\, ds \\
&\le |a| + \delta_0 L_1 + \delta_0 L_2 \\
&\le |a| + \frac{1}{1+L_1} L_1 + \frac{1}{1+L_2} L_2 \\
&\le |a| + 1 + 1 = |a| + 2 = R.
\end{aligned}$$

So $|y| \le R \implies |T(y)| \le R$, i.e.

$$T : \{\, y \in C([-\delta_0, \delta_0]) \text{ and } |y| \le R \,\}, \qquad T(y) \in C([-\delta_0, \delta_0]), \quad |T(y)| \le R.$$

Second Step

Let

$$M = \max_{|z| \le R} |f'(z)|, \qquad \delta = \min\left\{\delta_0,\ \frac{1}{2M+1}\right\},$$

and consider $T : \{\, y \in C([-\delta, \delta]) \mid |y| \le R \,\}$ with $y_1, y_2 \in T$. We want to show

$$d\big(T(y_1), T(y_2)\big) \le \tfrac{1}{2} d(y_1, y_2).$$

For $x \in [-\delta, \delta]$,

$$\begin{aligned}
T(y_1)(x) - T(y_2)(x) &= a + \int_0^x \big[f(y_1(s)) + g(s)\big]\, ds - \left(a + \int_0^x \big[f(y_2(s)) + g(s)\big]\, ds \right) \\
&= \int_0^x \big[f(y_1(s)) - f(y_2(s))\big]\, ds.
\end{aligned}$$

Let $z_1 = y_1(s)$ and $z_2 = y_2(s)$. Then

$$f(z_1) - f(z_2) = f'(z_3)\big(|z_1 - z_2|\big), \qquad \text{where } z_1 < z_3 < z_2.$$

Therefore

$$|f(y_1(s)) - f(y_2(s))| = |f'(z_3)|\,|y_1(s) - y_2(s)| \leq M\,|y_1(s) - y_2(s)|.$$