Lecture 22: Differentiating and Integrating Power Series; Ordinary Differential Equations

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1 Recap

Let $f_n, f : I \to \mathbb{R}$.

- **Pointwise convergence:** for each x fixed, $f_n(x) \to f(x)$.
- **Uniform convergence:** $\forall \varepsilon > 0$ $\exists N$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ $\forall x$.

If $f_n \in \mathcal{R}([a,b])$ and $f_n \to f$ uniformly, then $f \in \mathcal{R}([a,b])$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx.$$

2 Differentiation under Uniform Convergence

Theorem 1: *Let $f_n : I \to \mathbb{R}$ be differentiable, with $[a,b] \in I$ and $x_0 \in I$. Suppose*

1. $f_n(x_0) \to c$,
2. $f_n' \to g$ *uniformly,*
3. f_n' *continuous.*

Then there exists $f : I \to \mathbb{R}$, f is differentiable, f' is continuous, and

(a) $f_n \to f$ *uniformly,*

(b) $f_n' \to f'$ *uniformly.*

Proof. Since the f_n' are continuous and $f_n' \to g$ uniformly, g is also continuous. We can define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = c + \int_{x_0}^x g(s)\,ds.$$

By the FTC, f is differentiable with $f' = g$. We also have $f_n(x_0) \to c = f(x_0)$.

Part (b) follows from $f_n' \to g$ uniformly, with $g = f'$.

We need to show $f_n \to f$ uniformly, i.e. we must control $|f_n(x) - f(x)|$. Use the FTC to write $f_n(x)$ as

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(s)\,ds.$$

Then

$$\begin{aligned}
|f_n(x) - f(x)| &= \left| f_n(x_0) + \int_{x_0}^{x} f_n'(s)\,ds - \left(c + \int_{a}^{b} g(s)\,ds \right) \right| \\
&= \left| f_n(x_0) - c + \int_{x_0}^{x} f_n'(s)\,ds - \int_{x_0}^{x} g(s)\,ds \right| \\
&\le |f_n(x_0) - c| + \left| \int_{x_0}^{x} f_n'(s)\,ds - \int_{x_0}^{x} g(s)\,ds \right| \\
&= |f_n(x_0) - c| + \left| \int_{x_0}^{x} \big(f_n'(s) - g(s)\big)\,ds \right| \\
&\le |f_n(x_0) - c| + \int_{x_0}^{x} |f_n'(s) - g(s)|\,ds \\
&\le |f_n(x_0) - c| + (b-a)\,d(f_n', g).
\end{aligned}$$

As $n \to \infty$, this upper bound goes to zero, which gives convergence uniform in x. **Q.E.D.**

Example 2: Let

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \qquad E(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

By the Weierstrass M-test, for all $L > 0$, $E_n \to E$ uniformly on $[-L, L]$. Also

$$E_n'(x) = \sum_{k=0}^{n} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{n} \frac{ky^{k-1}}{k!} = \sum_{k=1}^{n} \frac{x^{k-1}}{(k-1)!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = E_{n-1}(x).$$

Then:

1. $E_n(0) = 1$ for each n,
2. $E_n' = E_{n-1} \to E$ uniformly on $[-L, L]$,
3. E_n' are polynomials $\implies$ continuity.

Therefore, by the theorem, on $[-L, L]$ there exists f with $E_n \to f$ uniformly and there exists f' with $E_n' \to f'$ uniformly, where $f = E(x)$. So $E_{n-1} \to E'$, but $E_{n-1} \to E$, and therefore

$$E' = E.$$

3 Power Series

For a power series $\sum_{n=0}^{\infty} a_n x^n$, the radius of convergence is given by

$$M = \limsup \sqrt[n]{|a_n|}, \qquad R = \frac{1}{M}.$$

We have

$$\sum_{k=0}^{n} a_k x^k \longrightarrow \sum_{k=0}^{\infty} a_k x^k \quad \text{uniformly on } [-L, L], \text{ where } L < R.$$

Define f by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

We want a formula for $f^{(m)}(x)$. Consider

$$\sum_{n=0}^{\infty} n a_n x^{n-1}$$

and its radius of convergence:

$$\sqrt[n]{n|a_n|} = n^{1/n} \sqrt[n]{|a_n|}, \qquad n^{1/n} = e^{\frac{\ln n}{n}} \to 1,$$

$$\limsup_{n\to\infty} \sqrt[n]{n|a_n|} = \limsup_{n\to\infty} \sqrt[n]{n} \cdot \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = M,$$

so $R_{\text{new}} = R_{\text{old}}$. Iterating this yields that the radius of convergence never changes.

$$\sum_{n=0}^{\infty} \prod_{k=0}^{m-1} (n-k)\, a_n x^{n-m} \qquad \text{or} \qquad \sum \frac{n!}{(n-m)!}\, a_n x^{n-m}.$$

Theorem 3: *Let $\sum a_n x^n$ have radius of convergence R, and let $0 < L < R$. Then on $[-L, L]$,*

$$\sum_{k=0}^{n} a_k x^k \longrightarrow \sum_{k=0}^{\infty} a_k x^k \quad \textit{uniformly}, \qquad \sum \frac{k!}{(k-m)!} a_k x^{k-m} \longrightarrow f^{(m)}(x) \quad \textit{uniformly}.$$

Proof. Follows from the earlier theorem. Assume first $m = 1$. Then

$$f_n(x) = \sum_{k=0}^{n} a_k x^k \longrightarrow f(x), \qquad f_n(0) = f(0),$$

$$f_n'(x) = \sum_{k=0}^{n} k a_k x^{k-1} \longrightarrow \sum_{k=0}^{\infty} k a_k x^{k-1} = f'.$$

Q.E.D.

4 Integrating Power Series

Consider $\int_a^b f(x)\, dx$ where

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad a, b \in (-R, R).$$

Since

$$f_n(x) = \sum_{k=0}^{n} a_k x^k \longrightarrow f(x) \quad \text{on } [-L, L] \text{ uniformly},$$

we have $\int_a^b f_n(s)\,ds \to \int_a^b f(s)\,ds$, and

$$\int_a^b f_n(s)\,ds = \int_a^b \sum_{k=0}^n a_k s^k\,ds = \sum \int_a^b a_k s^k\,ds.$$

If $a = 0$ and $b = x$, then $\int_0^x f_n(s)\,ds \to \int_0^x f(s)\,ds$, i.e.

$$\sum_{k=0}^n \int_0^y a_k x^k = \sum_{k=0}^n \frac{a_k}{k+1} x^{k+1} \longrightarrow \int_0^y f(s)\,ds.$$

5 Differential Equations

Definition 4: A *differential equation* is an equation in an unknown function and its derivative.

Example 5: $f'(x) = x$. The solutions are $f(x) = \frac{1}{2}x^2 + c$.

Ordinary Differential Equations (ODEs)

The unknown function will be denoted by y.

$$\begin{cases} y(0) = a, \\ y' = f(y) + g(x), \end{cases} \tag{$\oplus$}$$

where $f, g : \mathbb{R} \to \mathbb{R}$, f is differentiable and f' is continuous, and g is continuous.

Question: does this ODE have a solution? If so, is it unique?

We are looking for $y : I \to \mathbb{R}$ such that y' exists, $0 \in I$, and y satisfies $(\oplus)$.

Theorem 6 (Picard–Lindelöf)**:** *Given f, g as above, then for all a there exists $\delta > 0$ and a solution to $(\oplus)$ on $(-\delta, \delta)$. Moreover, it is unique.*

The proof uses the contracting mapping theorem.

Definition 7: Recall (X, d) and $T : X \to X$. Then T is said to be a *contracting map* if there exists $c < 1$ such that

$$|T(x) - T(y)| \leq c|x - y| \quad \forall x, y \in X.$$

Theorem 8: *If X is Cauchy complete and $T : X \to X$ is a contracting map, then T has a unique fixed point: there exists x_0 with $T(x_0) = x_0$.*