Lecture 21: Integrals and Derivatives under Uniform Convergence

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1 Two Notions of Convergence

Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$. There are two notions of convergence.

1. **Pointwise convergence:** $f_n \to f$, with x fixed.
2. **Uniform convergence:** for all x,

$$\forall \varepsilon > 0 \ \ \exists N \text{ such that } n \geq N \implies |f_n - f| < \varepsilon \quad \underline{\forall x}.$$

Example 1: Let $I = [0, 1]$ and

$$f_n(x) = x^n, \qquad f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise, but $f_n \not\to f$ uniformly.

Note: Pointwise convergence is not enough to guarantee that the limit of continuous functions is continuous.

2 Uniform Limits of Continuous Functions

Theorem 2: *Let $f_n, f : I \to \mathbb{R}$ with*

(a) f_n *are continuous,*

(b) $f_n \to f$ *uniformly.*

Claim 3: f *is also continuous.*

Proof. Given $\varepsilon > 0$, there exists N such that if $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in I.$$

This comes from the definition of uniform convergence.

Since f_N is continuous, then for any fixed x_0 there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3},$$

by the definition of continuity. So for $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Q.E.D.

Example 4: Let

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The $E_n(x)$ are all continuous, and by the Weierstrass M-test $E_n \to E$, where

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

is uniform on any fixed interval $[-L, L]$. By the theorem just proved, $E(x)$ is also continuous.

3 The Metric Space $C([a, b])$

Let $I = [a, b]$. Then $C([a, b])$ has a natural metric: for $f, g \in C([a, b])$,

$$d(f, g) = \max\{\, |f(x) - g(x)| \mid x \in I \,\}.$$

Proposition 5: *$f_n \to f$ uniformly $\iff$ $f_n \to f$ in the metric space $\big(C([a, b]), d\big)$.*

Proof.

$$|f(x) - f_n(x)| \leq \varepsilon \ \ \forall x \iff d(f_n, f) \leq \varepsilon \ \ \forall x.$$

Therefore if $f_n \to f$ uniformly, then $d(f, f_n) \to 0$, which holds if and only if $f_n \to f$ in the metric space. **Q.E.D.**

Theorem 6: *The metric space $\big(C([a, b]), d\big)$ is Cauchy complete.*

Proof. Let $f_n \in C([a, b])$ be a Cauchy sequence. Then there exists $f \in C([a, b])$ with $f_n \to f$ in the metric d.

We first construct f as follows. Fix $x \in [a, b]$. Then

$$|f_n(x) - f_m(x)| \leq \max_{z \in [a,b]} \big\{ |f_n(z) - f_m(z)| \big\}.$$

Claim 7: *This implies $f_n(x) \in \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$.*

This is the case because, given $\varepsilon > 0$, since f_n is a Cauchy sequence in $(C(I), d)$ there exists N such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq d(f_n, f_m) < \varepsilon.$$

Since $\mathbb{R}$ is Cauchy complete and $f_n(x)$ is a Cauchy sequence, $f_n(x) \to f(x)$, so $f_n \to f$ pointwise.

Given $\varepsilon > 0$, since f_n is a Cauchy sequence in $(C(I), d)$, there exists N such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall x.$$

Fix $n \geq N$ and let $m \to \infty$:

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \quad \forall x \implies \sup_{x \in I} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} = d(f_n, f) < \varepsilon.$$

This is the same as saying $f_n \to f$ uniformly, and therefore f is also continuous. **Q.E.D.**

Remark: $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ inside the radius of convergence.

4 Integration under Uniform Convergence

Theorem 8: *Let $[a, b]$ be given, $f_n \in \mathcal{R}([a, b])$, and $f_n \to f$ uniformly.*

Claim 9: *$f \in \mathcal{R}([a, b])$ and*

$$\int_a^b f \, dx = \lim_{n \to \infty} \int_a^b f_n \, dx.$$

Proof. We want to show $f \in \mathcal{R}([a, b])$. Use that all we need to show is: given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Since $f_n \to f$ uniformly, there exists N such that for $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}.$$

Let P have dividing points x_i, with subintervals $[x_{i-1}, x_i]$. Then

$$f(x) \leq f_n(x) + \big(f(x) - f_n(x)\big) \leq f_n(x) + |f(x) - f_n(x)| < f_n(x) + \frac{\varepsilon}{3(b-a)},$$

so

$$M_i^f = \sup_{[x_{i-1}, x_i]} f \leq \sup_{[x_{i-1}, x_i]} f_n(x) + \frac{\varepsilon}{3(b-a)} = M_i^{f_n} + \frac{\varepsilon}{3(b-a)}.$$

Likewise for the infimum, with $m_i^f = \inf_{[x_{i-1}, x_i]} f$ and $m_i^{f_n}$:

$$m_i^{f_n} - \frac{\varepsilon}{3(b-a)} \leq m_i^f \leq M_i^f \leq M_i^{f_n} + \frac{\varepsilon}{3(b-a)}. \qquad (*)$$

Recall
$$\sum_i m_i^f \, \Delta x_i = L(f,P), \qquad \sum_i M_i^f \, \Delta x_i = U(f,P).$$

Multiply $(*)$ by Δx_i and sum over i:

$$\sum_i \left(m_i^{f_n} - \frac{\varepsilon}{3(b-a)} \right) \Delta x_i \le \sum_i m_i^f \, \Delta x_i \le \sum_i M_i^f \, \Delta x_i \le \sum_i \left(M_i^{f_n} + \frac{\varepsilon}{3(b-a)} \right) \Delta x_i,$$

$$L(f_n,P) - \sum_i \frac{\varepsilon}{3(b-a)} \Delta x_i \le L(f,P) \le U(f,P) \le U(f_n,P) + \sum_i \frac{\varepsilon}{3(b-a)} \Delta x_i,$$

$$L(f_n,P) - \frac{\varepsilon}{3} \le L(f,P) \le U(f,P) \le U(f_n,P) + \frac{\varepsilon}{3}.$$

Let $n = N$. Since $f_N \in \mathcal{R}([a,b])$, there exists P with

$$U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{3}.$$

Therefore
$$U(f_n,P) + \frac{\varepsilon}{3} \le L(f_n,P) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Hence
$$|L(f,P) - U(f,P)| < \varepsilon,$$

which shows that $f \in \mathcal{R}([a,b])$.

We have now that

$$L(f_N,P) - \frac{\varepsilon}{3} \le L(f,P) \le \int_a^b f \, dx \le U(f,P) \le U(f_N,P) + \frac{\varepsilon}{3},$$

so

$$\boxed{\left| \int_a^b f \, dx - \int_a^b f_N \, dx \right| \le \varepsilon}$$

choosing N such that

$$U(f_N,P) - L(f_N,P) < \frac{\varepsilon}{3}, \qquad L(f_n,P) \le \int_a^b f_n \, dx \le U(f_n,P).$$

Q.E.D.

Example 10: Let
$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \qquad E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

By the Weierstrass M-test, $E_n \to E$ uniformly on any interval $[-L, L]$, so

$$\int_a^b E_n \, dx = \sum_{k=0}^{n} \int_a^b \frac{x^k}{k!} \, dx \longrightarrow \int_a^b E(x) \, dx.$$