# Lecture 20: Pointwise Convergence; Uniform Convergence

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## 1 Curve in the Plane (Arc Length)

Let $f, g : [a, b] \to \mathbb{R}$ be differentiable with continuous derivatives. Then $\gamma(s) = (f(s), g(s))$, $s \in [a, b]$, is a curve, and the length of $\gamma(s)$ is

$$\int_a^b \sqrt{(f'(s))^2 + (g'(s))^2}\, ds = L.$$

The curve runs from $\gamma = (f(a), g(a))$ to $\gamma = (f(b), g(b))$.

For a partition $a < c < b$ with subinterval $[x_{i-1}, x_i]$, the chord length is

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \begin{pmatrix} f(x_i) - f(x_{i-1}) \\ g(x_i) - g(x_{i-1}) \end{pmatrix} \right| = \sqrt{(f(x_i) - f(x_{i-1}))^2 + (g(x_i) - g(x_{i-1}))^2}.$$

By the MVT, $f(x_i) - f(x_{i-1}) = f'(c)\Delta x_i$ and $g(x_i) - g(x_{i-1}) = g'(c)\Delta x_i$. Now we just pull $\Delta x_i$ out of the square root and sum everything.

**Example 1:** $f(s) = s$, $g(s) = s^2$, $s \in [0, 1]$, so $\gamma(s) = (s, s^2)$ with $f'(s) = 1$, $g'(s) = 2s$. Then

$$L = \int_0^1 \sqrt{1 + (2s)^2}\, ds = \int_0^1 \sqrt{1 + 4s^2}\, ds.$$

## 2 Pointwise Convergence

Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$.

**Definition 2:** $f_n \to f$ pointwise if for all $x$ (fixed), $f_n(x) \to f(x)$.

**Example 3:** $f_n(x) = x^n$, $0 \le x \le 1$, and

$$f = \begin{cases} 0 & 0 \le x < 1, \\ 1 & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise. Fix $x$, assume $0 \le x < 1$: then $x^n \to 0$. At $x = 1$, $x^n = 1$.

**Example 4:** $\displaystyle\sum_{k=0}^{n} \frac{x^k}{k!} = f_n(x)$ with $f(x) = e^x$. Then $f_n(x) \to f(x)$ for all $x$ (fixed).

## 3 Uniform Convergence

**Definition 5:** Let $f_n, f : I \to \mathbb{R}$. Then $f_n \to f$ *uniformly* if for all $\varepsilon > 0$ there exists $N$ such that for $n \geq N$,

$$|f(x) - f_n(x)| < \varepsilon \quad \text{for all } x.$$

**Lemma 6:** *Uniform convergence* $\implies$ *pointwise convergence.*

**Example 7:** $f_n = x^n$ with

$$f(x) = \begin{cases} 0 & 0 \leq x < 1, \\ 1 & x = 1. \end{cases}$$

**Claim 8:** $f_n \not\to f$ *uniformly.*

By the intermediate value theorem, since each $f_n$ is continuous, there exists $x_n$ with $f_n(x_n) = \frac{1}{2}$. Then for $0 \leq x_n < 1$,

$$|f_n(x_n) - f(x_n)| = \left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2} \not< \varepsilon,$$

so there is no uniform convergence.

**Example 9:** $f_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$, $f(x) = e^x$. Then $f_n(x) \to f(x)$ converges uniformly on each bounded interval $[-L, L]$.

## 4 Weierstrass M-test

**Theorem 10** (Weierstrass M-test)**:** *Let* $f_n : I \to \mathbb{R}$ *with* $|f_n| \leq M_n$ *and* $\displaystyle\sum_{n=0}^{\infty} M_n < \infty$. *Then* $\displaystyle\sum_{k=0}^{n} f_k \to S$ *uniformly.*

*Proof.* Let $S_n(x) = \sum_{k=0}^{n} f_k(x)$. Want to show $S_n \to S$ uniformly. For $n > m$,

$$|S_n(x) - S_m(x)| = \left|\sum_{k=0}^{n} f_k(x) - \sum_{k=0}^{m} f_k(x)\right| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k.$$

Given $\varepsilon > 0$ there exists $N$ such that for $m \geq N$, $\sum_{k=m+1}^{n} M_k < \varepsilon$, so $S_n(x)$ is a Cauchy sequence for all $x$, and $S_n(x) \to S(x)$.

Moreover,

$$|S_n(x) - S_m(x)| \leq \sum_{k=m+1}^{n} M_k \leq \sum_{k=m+1}^{\infty} M_k.$$

Notice $\sum_{k=m+1}^{\infty} M_k$ does not depend on $n$, so as $n \to \infty$ the inequality still holds:

$$|S(x) - S_m(x)| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon, \qquad m \geq N.$$

**Q.E.D.**

**Example 11:** $f_n(x) = \dfrac{x^n}{n!}$, $S_n(x) = \sum_{k=0}^n f_k(x)$. Want to show $S_n(x) \to e^x$ uniformly on $[-L, L]$. Set $M_n = \dfrac{L^n}{n!}$; then $\sum_{n=0}^\infty \dfrac{L^n}{n!} < \infty$. For $x \in [-L, L]$,

$$|f_n(x)| = \frac{|x^n|}{n!} = \frac{|x|^n}{n!} \leq \frac{L^n}{n!} = M_n,$$

so

$$\sum_{k=0}^{n} \frac{x^k}{k!} \to e^x \quad \text{uniformly on } [-L, L].$$

# 5 Power Series and Radius of Convergence

For a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, with

$$\limsup_{n\to\infty} |a_n|^{1/n} = M,$$

the radius of convergence is $R = \dfrac{1}{M}$.

**Theorem 12:** *For $\sum a_n x^n$ with $R > L$,*

$$\sum_{k=0}^{n} a_k x^k \to \sum_{k=0}^{\infty} a_k x^k \quad \textit{uniformly on } [-L, L].$$