Lecture 2: Introduction to Real Numbers (Ctd.)

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1 Explorations into $\sqrt{2}$

What is the difference between $\mathbb{R}$ and $\mathbb{Q}$? The answer lies within $\sqrt{2}$. So, let us explore this mysterious number. What is it? Well, to analyze it, let us define $x \mid x = \sqrt{2}$. We can now analyze the variable x, which helps us think beyond what we have already conceived about $\sqrt{2}$. We know that $x \cdot x = x^2 = 2,\ (x > 0)$. And we can examine the following.

Claim: there does not exist an $x = \frac{m}{n}$ where $x^2 = 2$.

Proof by contradiction.

$$\text{Assume } \exists\ x = \frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{N}.$$

$$x^2 = 2 \Rightarrow \left(\frac{m}{n}\right)^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2.$$

We may assume that m and n do not have a common factor. If they did, we would just reduce the fraction down to the point where they do not.

Since $m^2 = 2n^2$, m has a factor of 2.

$$\text{Let } m = 2m_1 \Rightarrow m^2 = 4m_1^2 = 2n^2$$

$$2m_1^2 = n^2$$

This means that 2 is also a factor in n. $\lightning$.

Q.E.D.

Okay so now we know that $\sqrt{2}$ is not in the rationals, but it is definitely a number. So, how do we fit it into our number system? To do this, we define $\sqrt{2}$ as the limit of a sequence, which is something more advanced that will be covered later in the course.

2 Completeness

What is completeness? To define this best, we will use definitions of upper bounds.

Suppose $\mathbb{S}$ is an ordered set. Let us define $\mathbb{A} \mid \mathbb{A} \subseteq \mathbb{S}$. We now also define $M \mid M \geq a \; \forall \; a \in \mathbb{A}$. M is said to be an upper bound for $\mathbb{A}$. To help solidify this concept, consider

$$\mathbb{S} \in \mathbb{N}.$$

$$\mathbb{A} \subseteq \mathbb{S} \mid \mathbb{A} = \{1, 2, 3\}.$$

For this example, 4 is an upper bound for the subset of $\mathbb{A}$, and so is 5. However, 2 is not an upper bound since $2 \not\leq 3$. Another <u>very</u> important example is $\mathbb{Q}$ and $\mathbb{N}$. We know that $\mathbb{N} \subseteq \mathbb{Q}$. In this case, there is no valid upper bound. This will be proven later in this document. (See page 4, about the Archimedean property)

The least upper bound extends this further, stating that if $\mathbb{S}$ is an ordered set, and $\mathbb{A} \subseteq \mathbb{S}$ that is bounded, then M is a least upper bound for $\mathbb{A}$ if M is an upper bound and if any other upper bound, M_1, satisfies $M_1 \geq M$. In other words, a least upper bound is an upper bound that is the least. Completeness satisfies the condition that any bounded subset has a least upper bound.

There is a very important theorem that states $\exists$ a smallest complete ordered field that contains $\mathbb{Q}$. This theorem will not be proven due to its complex nature, but it is important to know. This complete ordered field that it is referencing is $\mathbb{R}$. Let us now show that $\sqrt{2} \in \mathbb{R}$. This proof is pretty long, and is the most complex we have seen to this point. So stick with me through it.

Claim: $\sqrt{2} \in \mathbb{R}$.

Proof:

$$\mathbb{A} = \{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 < 2\}.$$

$$\mathbb{A} \subseteq \mathbb{R}$$

$$\mathbb{A} \text{ is nonempty since } 1 \in \mathbb{A}.$$

$$\mathbb{A} \text{ is bounded.}$$

$$\forall \; x \in \mathbb{A}, x^2 < 2 \Rightarrow x < 2.$$

We define $\sqrt{2}$ to be the least upper bound of $\mathbb{A}$, and let this number be denoted by x. To prove the claim, we need to show that $x^2 = 2$. To do this, we will divide the proof into two inequalities.

1. $x^2 \leq 2$.

We will argue by contradiction. Assume not, that $x^2 > 2$. We will indicate a number, represented by $x - h$, where $1 > h > 0$.

$$(x-h)^2 = x^2 - 2xh + h^2 \geq x^2 - 2xh.$$

We want to show that for $h > 0$ there exists another upper bound for $\mathbb{A}$.

$$(x-h)^2 > 2 \Rightarrow x^2 - 2xh > 2.$$

$$x^2 - 2 > 2xh \Leftrightarrow h < \frac{x^2-2}{2x}.$$

$$\frac{x^2-2}{2x} \text{ is strictly positive.}$$

$$\therefore \exists\ h > 0. \lightning.$$

2. $x^2 \geq 2$.

We will, again, argue by contradiction. Assume not, $x^2 < 2$. We want to show that x is not an upper bound for $\mathbb{A}$. We will indicate a number, represented by $x + h$, where $1 > h > 0$.

$$x + h \in \mathbb{A}.$$

$$(x+h)^2 = x^2 + h^2 + 2hx.$$

$$\text{Since } h < 1, h^2 < h.$$

$$x^2 + h^2 + 2hx < x^2 + h + 2hx.$$

Here, we want to show that $(x+h)^2 < 2$, and that h exists.

$$x^2 + h + 2hx < 2 \Leftrightarrow h + 2hx < 2 - x^2 \Leftrightarrow h < \frac{2-x^2}{1+2x}.$$

$\frac{2-x^2}{1+2x}$ is strictly positive.

$$\therefore \exists\ h > 0. \lightning.$$

If both of these inequalities are true, this must mean that $x^2 = 2$.

Q.E.D.

Now with this proof, there is an interesting corollary. We have just proven that if a field, $\mathbb{F}$ is complete and contains the rationals and that $\sqrt{2} \in \mathbb{F}$, then this field cannot be the only the rationals. This is due to our earlier proof, that $\sqrt{2}$ is not a rational number. The interesting part is that this shows the rationals are <u>not complete</u>.

3 Archimedean property

The Archimedean property, in plain english, states that if you pick any real number, I can point to a natural number greater than the one you just picked. It seems a little straightforward, right? Well, let's prove it.

Claim: $\forall\ x \in \mathbb{R}, \exists\ n \in \mathbb{N} \mid n > x$.

Proof:

We will argue by contradiction. Suppose not. Then $\exists\ x \mid \forall\ n \in \mathbb{N}, n \leq x$. This means that x is an upper bound. By the least upper bound property, $\mathbb{N}$ is a bounded subset of $\mathbb{R}$. Let y be the least upper bound of $\mathbb{N}$.

$$\forall\ n \in \mathbb{N}, n \leq y.$$

The natural numbers are an ordered field.

$$n + 1 \leq y.$$

By induction, y cannot exist. $\lightning$.

Q.E.D.

Another interesting corollary comes from this. The corollary states that if you give me any two real numbers, I can tell you a rational number that sits in between them.

Claim: $x, y \in \mathbb{R},$ *where* $x < y$. $\exists\ \frac{m}{n} \mid x < \frac{m}{n} < y$.

Proof:

Let $\beta = y - x > 0$. By the Archimedean property $\exists\ n \in \mathbb{N} \mid n > \frac{1}{\beta} \Rightarrow \frac{1}{n} < \beta$. Observe $m \in \mathbb{Z}, m - 1 < xn$. Choose the largest m such that this condition is satisfied.

$$\frac{m-1}{n} \leq x.$$

Since m was the largest it could be, $y > \frac{m}{n} > x$.

Q.E.D.

This brings us to the end of lecture 2.