Lecture 19: Fundamental Theorem of Calculus

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1 Recap: Riemann Integral

Let $f : [a,b] \to \mathbb{R}$ bounded, P a partition $a = x_0 < x_1 < \cdots < x_n = b$, with $M_i = \sup_{[x_{i-1},x_i]} f$, $m_i = \inf_{[x_{i-1},x_i]} f$, $\Delta x_i = x_i - x_{i-1}$, and

$$U(f,P) = \sum M_i \Delta x_i, \qquad L(f,P) = \sum m_i \Delta x_i.$$

The upper and lower integrals are

$$\overline{S}_a^b f\,dx = \inf_P U(f,P), \qquad \underline{S}_a^b f\,dx = \sup_P L(f,P),$$

and $f \in R([a,b]) \iff \overline{S}_a^b f\,dx = \underline{S}_a^b f\,dx$.

2 Properties of the Integral

1. $f \in R([a,b])$, $c \in \mathbb{R} \implies cf \in R([a,b])$ and $\int_a^b (cf)\,dx = c\int_a^b f\,dx$.
2. $f, g \in R([a,b]) \implies f+g \in R([a,b])$ and $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$.
3. $f, g \in R([a,b])$, $f \le g \implies \int_a^b f\,dx \le \int_a^b g\,dx$.
4. $f \in R([a,b])$, $a < c < b \implies \int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$.

Lemma 1: *$f \in R([a,b]) \iff$ for all $\varepsilon > 0$ there exists P with $U(f,P) - L(f,P) < \varepsilon$.*

Proof of (2). Given $\varepsilon > 0$: since $f \in R([a,b])$ there exists P_1 with $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$; likewise since $g \in R([a,b])$ there exists P_2 with $U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$. Let P be the partition that has all the dividing points of P_1 and P_2 (a refinement of both). Then

$$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_1) < \tfrac{\varepsilon}{2},$$

$$U(g,P) - L(g,P) \le U(g,P_2) - L(g,P_2) < \tfrac{\varepsilon}{2}.$$

On a subinterval $[x_{i-1}, x_i]$,

$$m_i^f + m_i^g \le \inf_{[x_{i-1},x_i]} (f+g) \le f + g \le \sup_{[x_{i-1},x_i]} (f+g) \le \sup f + \sup g = M_i^f + M_i^g,$$

so

$$m_i^f + m_i^g \le m_i^{f+g} \le M_i^{f+g} \le M_i^f + M_i^g.$$

Therefore

$$\begin{aligned} U(f+g,P) - L(f+g,P) = M_i^{f+g}\Delta x_i - m_i^{f+g}\Delta x_i &\le (M_i^f + M_i^g)\Delta x_i - (m_i^f + m_i^g)\Delta x_i \\ &= M_i^f \Delta x_i - m_i^f \Delta x_i + M_i^g \Delta x_i + m_i^g \Delta x_i \end{aligned}$$

$= \varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > (U(f,P) - L(f,P)) + (U(g,P) - L(g,P))$. **Q.E.D.**

Proof of (4). Given $\varepsilon > 0$, since $f \in R([a,b])$ there exists P with $U(f,P) - L(f,P) < \varepsilon$. Let P^* be the refinement that, in addition to all the dividing points of P, also has c as a dividing point, so $U(f,P^*) - L(f,P^*) < \varepsilon$. Splitting at c,

$$U(f, P^* \cap [a,c]) + U(f, P^* \cap [c,b]) = U(f,P^*),$$

$$L(f, P^* \cap [a,c]) + L(f, P^* \cap [c,b]) = L(f,P^*),$$

so

$$\big(U(f,P^*\cap[a,c]) - L(f,P^*\cap[a,c])\big) + \big(U(f,P^*\cap[c,b]) - L(f,P^*\cap[c,b])\big) < \varepsilon.$$

Hence $\int_a^c f\,dx + \int_c^b f\,dx < \varepsilon$ (i.e. f is integrable on each piece and the integrals add). **Q.E.D.**

Corollary 2 (of (3))**:** *If $f, |f| \in R([a,b])$, then*

$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx.$$

Proof. $(*)$ $f \le |f|$ for all x; $(**)$ $-f \le |f|$ for all x. By $(*)$ and (3),

$$\int_a^b f\,dx \le \int_a^b |f|\,dx.$$

By $(**)$ and (3),

$$-\int_a^b f\,dx = \int_a^b (-f)\,dx \le \int_a^b |f|\,dx \implies -\int_a^b f\,dx \le \int_a^b |f|\,dx,$$

which gives the claim. **Q.E.D.**

3 Fundamental Theorem of Calculus, Version 1

How to evaluate integrals?

Theorem 3 (FTC, Version 1)**:** *Let f be continuous on $[a,b]$, and define $F(x) = \int_a^x f(s)\,ds$, $F : [a,x] \to \mathbb{R}$. Then F is differentiable and $F' = f$.*

Proof. Want to show F' exists at x_0. Consider $\dfrac{F(x)-F(x_0)}{x-x_0}$. Assume $x > x_0$. Then

$$\begin{aligned}F(x)-F(x_0) &= \int_a^x f(s)\,ds - \int_a^{x_0} f(s)\,ds \\ &= \int_a^{x_0} f(s)\,ds + \int_{x_0}^x f(s)\,ds - \int_a^{x_0} f(s)\,ds = \int_{x_0}^x f(s)\,ds.\end{aligned}$$

Bounding,

$$\int_{x_0}^x \inf_{[x_0,x]} f\,ds \le \int_{x_0}^x f(s)\,ds \le \int_{x_0}^x \sup_{[x_0,x]} f\,ds,$$

i.e.

$$\left(\inf_{[x_0,x]} f\right)(x-x_0) \le F(x)-F(x_0) \le \left(\sup_{[x_0,x]} f\right)(x-x_0),$$

so

$$\inf_{[x_0,x]} f \le \frac{F(x)-F(x_0)}{x-x_0} \le \sup_{[x_0,x]} f.$$

Take the limit as $x \to x_0$: since f is continuous,

$$\inf_{[x_0,x]} f \to f(x_0), \qquad \sup_{[x_0,x]} f \to f(x_0),$$

so the difference quotient is squeezed and must then also go to $f(x_0)$. **Q.E.D.**

4 Fundamental Theorem of Calculus, Version 2

Theorem 4 (FTC, Version 2)**:** *Let $F : [a,b] \to \mathbb{R}$ with $F' = f$ existing and $f \in R([a,b])$. Then*

$$F(b)-F(a) = \int_a^b f(x)\,dx.$$

(Proof skipped for time.)

Example 5:

$$\int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{3},$$

where the evaluation step uses FTC Version 2.

5 Improper Integrals

Two cases: (1) the interval is unbounded; (2) the function is unbounded.

Case 1: unbounded interval

$f \in R([a,b])$ for all $b > a$ (a fixed). Does $\int_a^\infty f\,dx$ make sense? Yes, if $\lim_{b\to\infty} \int_a^b f\,dx$ exists.

Case 2: unbounded function

$f \in R([c,b])$ for all $a < c < b$. Then $\int_a^b f(x)\,dx$ is defined if $\lim_{c\to a} \int_c^b f(x)\,dx$ exists.

Example 6: On $[1,\infty)$, $f(x) = \frac{1}{x^2}$. Does $\int_1^\infty \frac{1}{x^2}\,dx$ exist? With $F(x) = -\frac{1}{x}$, $F' = \frac{1}{x^2}$,

$$F(b) - F(1) = \int_1^b \tfrac{1}{x^2}\,dx = -\tfrac{1}{b} + 1 \to 1,$$

so the integral converges to 1.

Arc length can be defined as follows: for $\gamma(s) = (f(s), g(s))$,

$$L = \int_a^b \sqrt{(f'(s))^2 + (g'(s))^2}\,ds.$$