Lecture 18: Integrable Functions

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1 Riemann Integrals (recap)

Let $f : [a, b] \to \mathbb{R}$ and P a partition of $[a, b]$,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

with dividing points, subintervals $[x_{i-1}, x_i]$, and $\Delta x_i = x_i - x_{i-1}$. Restricting f to $[x_{i-1}, x_i]$, set

$$M_i = \sup_{[x_{i-1},x_i]} f, \qquad m_i = \inf_{[x_{i-1},x_i]} f,$$

and form the upper and lower sums

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Since $m_i \leq M_i$, we have $L(f, P) \leq U(f, P)$.

P_1 is a subpartition (refinement) of P if P_1 has the same dividing points as P and then some more.

2 Refinement Inequalities

For P_1 a refinement of P,

$$L(f, P_1) \geq L(f, P), \qquad U(f, P_1) \leq U(f, P).$$

When a new point y splits $[x_{i-1}, x_i]$ into $[y, x_i]$ (width $y - x_{i-1}$) and $[x_{i-1}, y]$ (width $x_i - y$),

$$\begin{aligned} L(f, P_1) &= \inf_{[y,x_i]} f\,(y - x_{i-1}) + \inf_{[x_{i-1},y]} f\,(x_i - y) \\ &\geq m_i(y - x_{i-1}) + m_i(x_i - y) = m_i(x_i - x_{i-1}) = m_i \Delta x_i. \end{aligned}$$

The same logic applies to the sup. So as we refine, L and U tend toward each other.

From here, we may turn the $\sum$ into an S (integral). f is said to be Riemann integrable ($f \in R([a, b])$) if

$$\underline{S}_a^b f\,dx = \overline{S}_a^b f\,dx.$$

In general,

$$\underline{S}_a^b f\,dx \leq \overline{S}_a^b f\,dx,$$

and importantly $L(f, P) \leq U(f, P^*)$ for any two partitions P, P^*.

Proof. P is a partition, P^* is a partition. Let $\bar{P}$ be the partition that has all the dividing points of both P and P^*, so $\bar{P}$ is a refinement of both. Then

$$L(f,P) \le L(f,\bar{P}) \le U(f,\bar{P}) \le U(f,P^*),$$

so $L(f,P) \le U(f,P^*)$. Taking sup/inf,

$$L(f,P) \le \inf_{P^*} U(f,P^*) = \overline{S}_a^b f\,dx, \qquad \underline{S}_a^b f\,dx = \sup_P L(f,P) \le \overline{S}_a^b f\,dx.$$

Q.E.D.

Example 1: $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & x \text{ rational}, \\ 0 & \text{otherwise}. \end{cases}$$

Claim 2: *$f(x)$ is not Riemann integrable.*

On any subinterval $[x_{i-1}, x_i]$, $M_i = 1$ and $m_i = 0$, so

$$U(f,P) = \sum M_i \Delta x_i = \sum \Delta x_i = 1, \qquad L(f,P) = \sum m_i \Delta x_i = 0.$$

3 The Cauchy Criterion for Integrability

Lemma 3: $f : [a,b] \to \mathbb{R}$. *Then $f \in R([a,b])$ if and only if for all $\varepsilon > 0$ there exists a partition P with*

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. If f is Riemann integrable,

$$\sup_P L(f,P) = \underline{S}_a^b f\,dx = \overline{S}_a^b f\,dx = \inf_P U(f,P).$$

Given $\varepsilon > 0$ there exists P_1 with $L(f,P_1) \ge \overline{S}_a^b f\,dx - \frac{\varepsilon}{2}$. Likewise there exists P_2 with $U(f,P_2) \le \overline{S}_a^b f\,dx + \frac{\varepsilon}{2}$. Let P be the partition of all the dividing points of P_1 and P_2. Then

$$L(f,P_1) \le L(f,P) \le U(f,P) \le U(f,P_2),$$

so

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\ &\le \Big(U(f,P_2) - \overline{S}_a^b f\,dx\Big) - \Big(L(f,P_1) - \overline{S}_a^b f\,dx\Big) \\ &\le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Q.E.D.

4 Continuous Functions are Integrable

Theorem 4: *Any continuous function on* $[a, b]$ *is Riemann integrable.*

In the proof we will need uniform continuity.

Definition 5: Let $f : I \to \mathbb{R}$. f is said to be *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

To understand the difference:

Example 6: $f(x) = x^2$ on $\mathbb{R}$. If f were uniformly continuous, then for a given $\varepsilon > 0$ there would be a single δ with $|f(x + \delta) - f(x)| < \varepsilon$ for all x. But

$$(x + \delta)^2 - x^2 = x^2 + \delta^2 + 2x\delta - x^2 = \delta^2 + 2x\delta,$$

which is not $< \varepsilon$ for all x (it grows with x).

Example 7: $f(x) = \frac{1}{x}$, $x \in (0, 1]$, is also not uniformly continuous.

Note: In my own words, a uniformly continuous function is one that allows any size δ to satisfy an ε; in other words, the function cannot have exponential growth.

Take $x_n = \frac{1}{n}$, $y_n = \frac{1}{2n}$. Then $|x_n - y_n| < \frac{1}{n}$, but with $f(x_n) = n$, $f(y_n) = 2n$,

$$|f(x_n) - f(y_n)| = n.$$

Theorem 8: *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* f *is uniformly continuous.*

Note: a compact interval is one that is bounded and closed.

Proof. Suppose not; then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in [a, b]$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. Use this for $\delta = \frac{1}{n}$, $n \in \mathbb{N}$: get $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon > 0$.

By the Bolzano–Weierstrass theorem, $x_{n_k} \to x$. Then

$$|y_{n_k} - x| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x| < \frac{1}{n_k} + |x_{n_k} - x| \to 0,$$

so $y_{n_k} \to x$ as well. Since f is continuous, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. However, now for the fixed $\varepsilon > 0$, the statement $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ cannot be satisfied – a contradiction. **Q.E.D.**

Proof that continuous $\implies$ *Riemann integrable.* We showed earlier that we need only show: given $\varepsilon > 0$, there exists P with $U(f, P) - L(f, P) < \varepsilon$. Given $\varepsilon > 0$, since f is uniformly continuous, there exists $\delta > 0$ such that if $|x - y| < \delta$ then

$$|f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let P be a partition where each $\Delta x_i \leq \delta$. On $[x_{i-1}, x_i]$ (of width at most δ), with $M_i = \sup f = f(y)$ and $m_i = \inf f = f(y^*)$,

$$M_i - m_i = f(y) - f(y^*) < \frac{\varepsilon}{b-a}.$$

Then

$$U(f,P) - L(f,P) = \sum M_i \Delta x_i - \sum m_i \Delta x_i = \sum (M_i - m_i) \Delta x_i < \frac{\varepsilon}{b-a} \sum \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Q.E.D.

5 Properties of the Integral

Theorem 9: *1. If $f \in R([a,b])$ and $c \in \mathbb{R}$, then $cf \in R([a,b])$ and*

$$\int_a^b (cf)\, dx = c \int_a^b f\, dx.$$

2. If $f, g \in R([a,b])$, then $f + g \in R([a,b])$ and

$$\int_a^b (f+g)\, dx = \int_a^b f\, dx + \int_a^b g\, dx.$$

3. If $f, g \in R([a,b])$ with $f \leq g$, then

$$\int_a^b f\, dx \leq \int_a^b g\, dx.$$

4. If $f \in R([a,b])$ and $a < c < b$, then $f \in R([a,c])$ and $f \in R([c,b])$, with

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^d f\, dx.$$