Lecture 17: Taylor Polynomials; Remainder Term; Riemann Integrals

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1 Two Mean Value Inequalities

1. (MVT) $f : [a, b] \to \mathbb{R}$, f differentiable. There exists x_0 with $a < x_0 < b$ and
$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$
2. (Cauchy MVT) $f, g : [a, b] \to \mathbb{R}$ differentiable. There exists x_0 with $a < x_0 < b$ and
$$f'(x_0)(g(b) - g(a)) = g'(x_0)(f(b) - f(a)).$$

Observation. Take $g(x) = x$, so $g'(x) = 1$, with $f : [a, b] \to \mathbb{R}$ differentiable. By Cauchy MVT there exists x_0, $a < x_0 < b$, with

$$f'(x_0)(b - a) = f(b) - f(a) \implies f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

The Cauchy MVT is more general.

2 L'Hôpital's Rule

Let $f, g : (a, b) \to \mathbb{R}$ differentiable, with $g(x), g'(x) \neq 0$ for all x, and

$$\lim_{x \to a} g(x) \neq 0 \neq \lim_{x \to a} f(x).$$

Assume also that $\lim_{x \to a} \dfrac{f'(x)}{g'(x)} = L$.

Claim 1: $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Proof. Apply the Cauchy MVT on $a < y < x < b$:

$$f'(x_0)(g(x) - g(y)) = g'(x_0)(f(x) - f(y)),$$

so

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f(x) - f(y)}{x - y} \cdot \frac{x - y}{g(x) - g(y)}.$$

Since $\lim_{z\to a} \dfrac{f'(z)}{g'(z)} = L$: given $\varepsilon > 0$, there exists $\delta > 0$ with $a < z < a + \delta$ and

$$\left| \frac{f'(z)}{g'(z)} - L \right| < \varepsilon.$$

If $a < x < a + \delta$, since $a < x_0 < x \implies a < x_0 < a + \delta$. **Q.E.D.**

3 Taylor Expansion and Remainder

Let $f : [a, b] \to \mathbb{R}$ be k times differentiable (f', f'', $\dots$, $f^{(k)}$ exist). With

$$P_{k-1}(x) = \sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}(x-a)^i,$$

we have

$$f(x) = P_{k-1}(x) + \underbrace{\frac{f^{(k)}(c)}{k!}(x-a)^k}_{\text{remainder term}}, \qquad a < c < x.$$

How well does the Taylor polynomial approximate a function? The remainder term is denoted $R_k(c)$, where $a < c < x$ is all we know about c.

Example 2: $f(x) = e^x$, $f : [0, 1] \to \mathbb{R}$. Then $f'(x) = e^x$, $\dots$, $f^{(k)}(x) = e^x$, and with $a = 0$,

$$P_{k-1} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^{k-1}}{(k-1)!}.$$

Then

$$|P_{k-1}(x) - f(x)| = R_k(c) = \frac{e^c}{k!}x^k,$$

so

$$|R_k(c)| \le \frac{e}{k!},$$

a good approximation.

Example 3:

$$f(x) = \begin{cases} 0 & x \le 0, \\ e^{-1/x^2} & x > 0. \end{cases}$$

f is infinitely differentiable. Taylor expansion at 0: $P_{k-1}(x) = 0$ and $f(0) = 0$, so $f(x) = R_k(c)$.

Note: Side: $|R_k(c)| \le \dfrac{e^x}{k!}x^k < \varepsilon$.

4 Riemann Integrals

Let $f : [a, b] \to \mathbb{R}$ with f bounded.

Idea. Define area below the graph. The only thing you know: for a rectangle of width d and height e, area $= d \cdot e$.

Partitions

Definition 4: A partition P of $[a, b]$ is finitely many dividing points

$$a = x_0 < x_1 < \cdots < x_k = b,$$

dividing the original interval into subintervals $[x_{i-1}, x_i]$.

Definition 5: A *subpartition* (aka refinement) P_2 of a partition P_1 is a partition that has all the dividing points of P_1 and then some.

Let $f : [a, b] \to \mathbb{R}$ be bounded and P a partition of $[a, b]$. We define upper/lower sums $U(f, P)$ and $L(f, P)$. For a subinterval $[x_1, x_2]$,

$$U(f,P) : \left(\sup_{[x_1,x_2]} f\right)(x_2 - x_1), \qquad L(f,P) : \left(\inf_{[x_1,x_2]} f\right)(x_2 - x_1).$$

With $\Delta x_i = x_i - x_{i-1}$, $M_i = \sup_{[x_{i-1},x_i]} f$, $m_i = \inf_{[x_{i-1},x_i]} f$,

$$U(f,P) = \sum_{i=1}^{k} M_i \Delta x_i, \qquad L(f,P) = \sum_{i=1}^{k} m_i \Delta x_i.$$

Lemma 6: $L(f, P) \leq U(f, P)$.

Example 7: $f(x) = x^2 + 1$, $f : [-2, 2] \to \mathbb{R}$, $P = \{-2, -1, 0, 1, 2\}$ with $x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$, $\Delta x = x_i - x_{i-1}$. Then

$$M_1 = \sup_{[x_0,x_1]} f = \sup_{[-2,-1]} f = 5, \quad M_2 = 2, \quad M_3 = 2, \quad M_4 = 5,$$

$$m_1 = 2, \quad m_2 = 1, \quad m_3 = 1, \quad m_4 = 2.$$

So

$$U(f,P) = 5 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 5 \cdot 1 = 14, \qquad L(f,P) = 6.$$

5 Refinement Lemma and Upper/Lower Integrals

Lemma 8: *Let $f : [a, b] \to \mathbb{R}$ bounded, P_1 a partition, P_2 a subpartition of P_1. Then*

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1).$$

Proof. On a subinterval $[x_{i-1}, x_i]$ split by a new point y,

$$\sup_{[x_{i-1},y]} f \leq \sup_{[x_{i-1},x_i]} f,$$

and similarly for the other pieces; extend the logic to the other cases. **Q.E.D.**

Upper and Lower Integrals

Basically turn $\sum M_i \Delta x$ into $\overline{S}_a^b f\,dx$ (upper) and $\sum m_i \Delta x$ into $\underline{S}_a^b f\,dx$ (lower).

Definition 9: Let $f : [a, b] \to \mathbb{R}$ be bounded. We say it is *Riemann integrable* if $\underline{S} = \overline{S}$, and if that is the case we denote it $\int_a^b f(x)\,dx$.

Note that $L(f, P) \leq U(f, P)$ and

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1),$$

so

$$\underline{S}_a^b f(x)\,dx \leq \overline{S}_a^b f(x)\,dx.$$