Lecture 16: Rolle's Theorem; Mean-Value Theorem; Taylor Expansion

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1 Recap

Let $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$. If f is differentiable at x_0, then $\dfrac{f(x) - f(x_0)}{x - x_0}$ has a limit as $x \to x_0$.

Lemma 1: *If f is differentiable at x_0, it is also continuous at x_0.*

We also had the four derivative rules, and the local max–min definition.

Let $f : [a, b] \to \mathbb{R}$ be differentiable. We say a max exists if there exists $\delta > 0$ such that

$$\max f\big|_{[x_0-\delta, x_0+\delta]} = \max f.$$

This also applies, with obvious changes, to a minimum.

2 Critical Points

Lemma 2: *If $f : [a, b] \to \mathbb{R}$ is differentiable and has a local maximum/minimum at $x_0 \in (a, b)$, then $f'(x_0) = 0$.*

Proof. Assume that f has a local maximum at x_0. For $x > x_0$ and f sufficiently close,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Now take $x < x_0$ and sufficiently close,

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Thus $f'(x_0) = 0$. **Q.E.D.**

Critical points are where $f'(x_0) = 0$.

Example 3: $f(x) = x^2$, $f'(x) = 2x$, so $x = 0$ is a critical point.

3 Rolle's Theorem

Theorem 4 (Rolle's theorem)**:** *Let $f : [a, b] \to \mathbb{R}$ be differentiable with $f(a) = f(b)$. Then there exists x_0 where $a < x_0 < b$ and $f'(x_0) = 0$.*

Proof. Three cases to consider:

1. $f \equiv f(a)$;
2. $\exists x \mid f(x) > f(a) = f(b)$;
3. $\exists x \mid f(x) < f(a) = f(b)$.

For case (1) we already know $f' = 0$ everywhere. For case (2), by the extreme value theorem f attains a maximum somewhere in the interior (it has to come back down to $f(b)$), and $f' = 0$ there. Same for case (3). **Q.E.D.**

4 Mean Value Theorem

First consequence of Rolle's theorem:

Theorem 5 (Mean Value Theorem)**:** *Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then there exists x where $a < x < b$ and*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Set

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then

$$h(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a),$$

$$h(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

So $h(a) = h(b)$. By Rolle's theorem, $h'(x) = 0$ for some x where $a < x < b$, and

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} = 0.$$

Q.E.D.

5 Cauchy Mean Value Theorem

Theorem 6 (Cauchy Mean Value Theorem)**:** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable. Then there exists x_0 with $a < x_0 < b$ and*

$$f'(x_0)(g(b) - g(a)) = g'(x_0)(f(b) - f(a)).$$

Observe that if $g(b) \neq g(a)$ and $g'(x_0) \neq 0$,

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(b) - f(a)}{b - a} \cdot \frac{b - a}{g(b) - g(a)}.$$

Proof. Set
$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$
Then
$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)).$$
Want to find x_0 with $h'(x_0) = 0$.
$$h(a) = f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) = f(a)g(b) - g(a)f(b),$$
$$h(b) = f(b)(g(b) - g(a)) - g(b)(f(b) - f(a)) = -g(a)f(b) + g(b)f(a),$$
so $h(a) = h(b)$. By Rolle's theorem we can now use the same method as we did in MVT to prove this. **Q.E.D.**

6 Taylor Expansion (Special Case)

Let $f : \mathbb{R} \to \mathbb{R}$ differentiable with $f' : \mathbb{R} \to \mathbb{R}$ also differentiable.

Idea. We want to say that f locally can be approximated by a polynomial, for f near x_0. The Taylor polynomial at x_0:
$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$
the first-order Taylor expansion at x_0. Idea: $f(x) \approx P_1(x)$ if $x \approx x_0$.

Theorem 7: *There exists c between a and b ($a < c < b$) such that*
$$f(b) = P_1(b) + \frac{f^{(2)}(c)}{2}(b - a)^2.$$

Proof. Define M by
$$f(b) - P_1(b) = \frac{M}{2}(b - a)^2,$$
i.e. by definition $f(b) = P_1(b) + \frac{M}{2}(b - a)^2$. Need to show $M = f''(c)$. Set
$$h(x) = f(x) - P_1(x) - \frac{M}{2}(x - a)^2.$$
Then
$$h(b) = f(b) - P_1(b) - \frac{M}{2}(b - a)^2 = 0 \quad \text{(by definition of } M\text{)},$$
$$h(a) = f(a) - P_1(a) = 0.$$
By Rolle's theorem there exists c_1 between a and b with $h'(c_1) = 0$. Now
$$h'(x) = f'(x) - f'(a) - M(x - a),$$
so $h'(a) = 0$ and $h'(c_1) = 0$. By Rolle's theorem a second time, there exists $c_2 = c$ with $h''(c) = 0$. Since
$$h''(x) = f''(x) - M, \qquad 0 = h''(c) = f''(c) - M \implies M = f''(c).$$

Q.E.D.

7 Taylor Expansion (General Case)

Let $f : \mathbb{R} \to \mathbb{R}$ and assume f is differentiable all the way up to $f^{(k)}$ (which exists). On $[a, b]$, define the Taylor polynomial

$$
\begin{aligned}
P_{k-1}(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} \\
&= \sum_{i=0}^{k-1} \frac{f^{(i)}(a)}{i!}(x-a)^i.
\end{aligned}
$$

Theorem 8 (Taylor Expansion)**:** *Let $f : \mathbb{R} \to \mathbb{R}$. Then*

$$f(b) = P_{k-1}(b) + \frac{f^{(k)}(c)}{k!}(b-a)^k,$$

where $a < c < b$.

Proof. Define M by

$$f(b) = P_{k-1}(b) + \frac{M}{k!}(b-a)^k.$$

Set

$$h(x) = f(x) - P_{k-1}(x) - \frac{M}{k!}(x-a)^k.$$

By definition of M, $h(b) = 0$, and

$$h(a) = f(a) - P_{k-1}(a) - \frac{M}{k!}(a-a)^k = 0.$$

For $a < c_1 < b$: $h'(c_1) = 0$ and $h'(a) = 0$, which gives $a < c_2 < c_1$, etc. **Q.E.D.**