Lecture 15: Derivatives; Laws for Differentiation

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1 The Derivative

Definition 1: Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and we call it $f'(x_0)$. The quantity inside the limit is called the *difference quotient* (assuming $x \neq x_0$).

Example 2: $f = c$ for some constant $c \in \mathbb{R}$:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{c - c}{x - x_0} = 0,$$

so f is differentiable with $f' = 0$.

Example 3: $f(x) = x$:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \frac{x - x_0}{x - x_0} = 1,$$

so f is differentiable with $f' = 1$.

In general, how do we compute the derivative? Laws:

1. Sum rule.
2. Leibniz rule.
3. Quotient rule.
4. Chain rule.

Throughout, $f, g : \mathbb{R} \to \mathbb{R}$ and both f, g are differentiable at x_0.

2 Sum Rule

Lemma 4 (Sum rule)**:** *$(f+g)(x) = f(x) + g(x)$ is differentiable at x_0, and*

$$(f+g)'(x) = f'(x) + g'(x).$$

Proof. Form the difference quotient:

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) + g(x) - (f(x_0) + g(x_0))}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \implies f'(x_0) +$$

Q.E.D.

3 Differentiability Implies Continuity

Lemma 5: *Let $f : \mathbb{R} \to \mathbb{R}$. If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Since $\dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon.$$

Take $\varepsilon = 1$:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq |x - x_0|,$$

so

$$|f(x) - f(x_0)| < |x - x_0| + |f'(x_0)|\,|x - x_0| = |x - x_0|(1 + |f'(x_0)|).$$

Given $\varepsilon > 0$, let $\delta = \min\left\{\delta_1, \dfrac{\varepsilon}{1 + |f'(x_0)|}\right\}$. For $|x - x_0| < \delta_1$, $|f(x) - f(x_0)| < |x - x_0|(1 + |f'(x_0)|)$ holds. **Q.E.D.**

4 Examples: Non-differentiable and Differentiable Cases

Example 6: $f(x) = |x|$, $f : \mathbb{R} \to \mathbb{R}$. f is continuous. The difference quotient at $x_0 = 0$:

$$x > 0 : \quad \frac{f(x) - f(0)}{x - 0} = \frac{x - 0}{x - 0} = 1; \qquad x < 0 : \quad f(x) = -x, \ \frac{f(x) - 0}{x - 0} = \frac{-x}{x} = -1.$$

The one-sided limits don't agree, so the limit doesn't exist (f is not differentiable at 0).

Example 7: $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

f is continuous at $x_0 = 0$. The difference quotient:

$$\frac{f(x) - f(0)}{x - 0} = \frac{x \sin \frac{1}{x} - 0}{x - 0} = \sin \tfrac{1}{x},$$

which fluctuates between -1 and 1 as $x \to 0$ (so f is not differentiable at 0).

Example 8: $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then

$$\frac{f(x) - 0}{x - 0} = \frac{x^2 \sin \frac{1}{x}}{x} = x \sin \frac{1}{x} \to 0 \text{ as } x \to 0,$$

so f is differentiable at 0.

5 Leibniz Rule

Lemma 9 (Leibniz rule)**:** *$(fg)(x) = f(x)g(x)$, and*

$$(fg)'(x) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Proof.

$$\begin{aligned} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= g(x)\frac{f(x) - f(x_0)}{x - x_0} + f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \\ &\implies g(x)f'(x) + f(x)g'(x). \end{aligned}$$

Q.E.D.

6 Quotient Rule

Lemma 10 (Quotient rule)**:** *$f, g : \mathbb{R} \to \mathbb{R}$ differentiable at x_0, $g(x) \neq 0$. Then $\left(\frac{f}{g}\right)$ is differentiable at x_0, and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof.

$$\begin{aligned} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} &= \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0} = \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\ &= \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\ &= g(x_0)\frac{f(x) - f(x_0)}{(x - x_0)}\frac{1}{g(x)g(x_0)} + f(x_0)\frac{g(x_0) - g(x)}{(x - x_0)}\frac{1}{g(x)g(x_0)} \\ &= \frac{g(x_0)f'(x_0)}{g(x)g(x_0)} - \frac{f(x_0)g'(x_0)}{g(x)g(x_0)}. \end{aligned}$$

But remember we took $\lim_{x \to x_0}$, so

$$= \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Q.E.D.

7 Chain Rule

Definition 11: For $f : (a,b) \to (c,d)$ and $g : (c,d) \to \mathbb{R}$ with $x \in (a,b)$,

$$(g \circ f)(x) = g(f(x)).$$

Theorem 12 (Chain rule)**:** *If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then*

$$(g(f(x_0)))' = g'(f(x_0)) \cdot f'(x_0).$$

Proof. Assume first $f'(x_0) \neq 0$. Since $\dfrac{f(x) - f(x_0)}{x - x_0} \to f'(x_0) \neq 0$, we have $f(x) \neq f(x_0)$ for x sufficiently close to x_0. Writing $y = f(x)$, $y_0 = f(x_0)$,

$$\begin{aligned}
\frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \frac{g(y) - g(y_0)}{y - y_0} \cdot \frac{y - y_0}{x - x_0} \\
&= g'(y_0) \cdot \frac{f(x) - f(x_0)}{x - x_0} \\
&= g'(y_0) \cdot f'(x_0) = g'(f(x_0)) \cdot f'(x_0).
\end{aligned}$$

Now assume $f(x) - f(x_0) = 0$. Observe

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0}, \qquad \text{since } f(x) - f(x_0) = 0,\ f(x) = f(x_0).$$

This trivializes the difference quotient, so $(g(f(x_0)))' = 0$ in this case, and the chain rule always holds. **Q.E.D.**

8 Critical Points

Lemma 13: *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If x_0 is a local max or local min for f, then $f'(x_0) = 0$.*

Local max at x_0 means there exists $\delta > 0$ such that on $(x_0 - \delta, x_0 + \delta)$,

$$f(x_0) = \max_{x \in (x_0 - \delta, x_0 + \delta)} f.$$

(Will be proved next time.)