Lecture 14: Sequential Compactness; Bolzano–Weierstrass Theorem in a Metric Space

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1 Covers and Compactness (recap)

Let (X, d) be a metric space. A *cover* of X is a collection of subsets U_α with $X \subseteq \bigcup_\alpha U_\alpha$. An *open cover* is a cover that consists of open subsets. We say that $A \subseteq X$ is *compact* if every open cover of A has a finite subcover.

Theorem 1: *Let (X, d), $A \subseteq X$. If A is compact, then A is bounded and closed. In $\mathbb{R}^n$ with the usual Euclidean metric the converse is also the case, but in general not.*

Example 2: $C([0, 1])$, $f, g \in C([0, 1])$, $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$. The closed ball

$$\{f \in C([0, 1]) \mid d(f, 0) \leq 1\} = \bar{B}_1(0)$$

is closed. For each $n \in \mathbb{N}$, define $f_n \in C([0, 1])$ by

$$f_n = \begin{cases} 1 & 0 \leq x \leq \frac{1}{n+1}, \\ 1 - n(n+1)\left(x - \frac{1}{n+1}\right) & \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ 0 & \frac{1}{n} \leq x \leq 1. \end{cases}$$

Then

$$d(f_n, f_{n+1}) = \max_x |f_n(x) - f_{n+1}(x)| = 1,$$

so $f_{n+1} \notin B_{1/2}(f_n)$. With $\bar{B}_1(0) = \bigcup_{f\in\bar{B}_1(0)} B_{1/2}(f)$, using the properties of f_n it follows that finitely many of these balls do not cover $\bar{B}_1(0)$.

Note: This makes zero sense, I do not understand what is above or how it relates to the topics of this lecture.

2 Bolzano–Weierstrass in a Metric Space

The Bolzano–Weierstrass theorem for a metric space holds when (X, d) is compact.

Theorem 3: *Let (X, d) be a metric space. If $A \subseteq X$ is compact, then any sequence in A has a convergent subsequence.*

Lemma 4: *Let (X, d), $A \subseteq X$ compact. Let C_i be a nested sequence of closed subsets of A (i.e. $C_i \subseteq X$, $\{C_i\}_{i=1,2,\ldots}$ with $C_{i+1} \subseteq C_i$). Assume also that $C_i \neq \emptyset$.*

Claim 5: $\bigcap_i C_i \neq \emptyset$.

Proof. Since C_i is closed, $B_i = X \setminus C_i$ is open, and $B_i \subseteq B_{i+1}$. If $\bigcap_i C_i = \emptyset$, then

$$A \subseteq \bigcup_i \underbrace{(X \setminus C_i)}_{B_i}.$$

Since A is compact,

$$A \subseteq \bigcup_\alpha B_\alpha \subseteq B_N = (X \setminus C_N),$$

so $C_N \cap A = \emptyset$. Since $C_N \subseteq A$, this gives $C_N = C_N \cap A = \emptyset$, a contradiction. **Q.E.D.**

Corollary 6: *Let (X, d) be a metric space, $A \subseteq X$ compact. Let $C_i = \bar{B}_{r_i}(x_i)$ with $\bar{B}_{r_{i+1}}(x_{i+1}) \subseteq \bar{B}_{r_i}(x_i)$ and $r_i \to 0$.*

Claim 7: $\bigcap_i \bar{B}_{r_i}(x_i) = \{x\}$.

Proof. $\bigcap_i \bar{B}_{r_i}(x_i) \neq \emptyset$ follows from the lemma. **Q.E.D.**

Note: This proof is pretty trivial and obvious.

3 Idea of the Proof of the Metric B–W Theorem

Let (X, d), $A \subseteq X$ compact, and x_n a sequence contained in A. Want to find a convergent subsequence.

Before the proof. For $r > 0$, $A \subseteq \bigcup_{y \in A} B_r(y)$, so finitely many cover A since A is compact:

$$A \subseteq B_r(y_1) \cup \cdots \cup B_r(y_m).$$

For one of these balls there exist infinitely many elements of the sequence – meaning for one of these balls there are infinitely many n such that $x_n \in B_r(y_j)$. Therefore there exists a subsequence of the given sequence that is contained in this ball.

Recall: (X, d) a metric space, A a compact subset, B a closed subset of X. Then

$$\underbrace{A \cap B}_{\text{compact} \,\cap\, \text{closed}} \subseteq A$$

is compact. Cover $\bar{B}_r(y_i)$ by balls of radius $\frac{r}{4}$; then $\bar{B}_r(y_i) \cap A$ is compact.

The scheme:

1. A is compact.

2. $B_r(y_j)$ contains a subsequence of x_n.

3. $\bar{B}_r(y_j) \cap A$ is compact.

4. Cover it by finitely many balls of radius $\frac{r}{4}$.

5. A subsequence of the subsequence lies in one of these finitely many balls of radius $\frac{r}{4}$.

Iterating,

$$A \rightsquigarrow B_1(y_j) \rightsquigarrow B_{r/4}(z_i), \qquad \bar{B}_{r/2}(z_1) \subseteq \bar{B}_{2r}(y_j).$$

Proof. $A \subseteq X$, $\{x_n\} \subseteq A$, $r = 1$. Successively covering by balls of shrinking radius gives a nested sequence of closed balls

$$\bar{B}_2 \supseteq \bar{B}_{1/2} \supseteq \bar{B}_{1/8} \supseteq \cdots$$

Q.E.D.