Lecture 13: Open & Closed Sets; Coverings; Compactness

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1 Open Sets (recap)

Let (X, d) be a metric space.

Definition 1: $O \subseteq X$ is said to be open if for all $x \in O$ there exists $r > 0$ such that $B_r(x) \subseteq O$.

Theorem 2: *Let (X, d), $x \in X$, $r > 0$. With $B_r(x) = \{y \in X \mid d(x, y) < r\}$, the ball $B_r(x)$ is open. ($B_r(x)$ is a ball of radius r centered at x.)*

On subsets there are three operations: (1) intersection, (2) union, (3) complement.

Lemma 3: *Let (X, d). If O_α is a family of open subsets, then $\bigcup_\alpha O_\alpha$ is also open. (Proof in previous notes.)*

Lemma 4: *Let (X, d) and $O_1, \ldots, O_n$ finitely many open subsets. Then $\bigcap_n O_n$ is also open.*

Example 5: $X = \mathbb{R}$, $O_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in \mathbb{N}$. Then $\bigcap_n O_n = \{0\}$. Note: each O_n is open, but $\bigcap_n O_n$ is not open.

Lemma 6: *Let (X, d), $r > 0$, $x \in X$. Then*

$$E = \{y \in X \mid d(x, y) > r\}$$

is an open subset.

Proof. Let $y \in E$, so $d(y, x) > r$. Set $s = d(x, y) - r > 0$. Take $z \in B_s(y)$; want to estimate $d(x, z)$ from below. By the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + s,$$

so

$$d(x, y) - s < d(x, z) \implies r < d(x, z).$$

Q.E.D.

2 Closed Sets

Definition 7: A subset $C \subseteq X$ is closed if $X \setminus C$ is open.

Since $\emptyset$ and X are both open, X and $\emptyset$ are also both closed.

Theorem 8: *Let (X, d). $C \subseteq X$ is closed $\iff$ for all convergent sequences x_n with $x_n \in C$, the limit x is also in C.*

Proof. Assume that C is closed and let $x_n \in C$ be a convergent sequence. Want to show that $x_n \to x$ is also in C. Since $x_n \to x$, for all $r > 0$ there exists $x_n \in B_r(x)$. If $x \notin C$, then since C is closed, $X \setminus C$ is open, so there exists $r > 0$ with $B_r(x) \subseteq X \setminus C$ – a contradiction, since $x_n \in C \cap B_r(x)$.

We also need to show the converse: for all sequences $x_n \in C$ that converge, if the limit x is also in C, then C is closed. Need to show $X \setminus C$ is open. Take $x \in X \setminus C$; need to show there exists $r > 0$ with $B_r(x) \subseteq X \setminus C$. If this were not the case, then for all $n \in \mathbb{N}$ choose $x_n \in B_{1/n}(x)$ with $x_n \in C$ and

$$d(x_n, x) < \tfrac{1}{n} \implies x_n \to x \implies x \in C,$$

a contradiction. **Q.E.D.**

Example 9: Closed subsets of $\mathbb{R}$: $[0,1]$ is closed; $[0,1)$ is neither closed nor open; $[0,1] \cup [2,3]$ is also closed; $\{0\}$ is closed.

Theorem 10: *Let (X, d) and C_α a family of closed subsets. Then $\bigcap_\alpha C_\alpha$ is also closed. If $C_1, \dots, C_n$ are closed, then $C_1 \cup \dots \cup C_n$ is also closed.*

Proof. By De Morgan,

$$X \setminus \bigcap_\alpha C_\alpha = \bigcup_\alpha (X \setminus C_\alpha), \qquad X \setminus (C_1 \cup \dots \cup C_n) = (X \setminus C_1) \cap \dots \cap (X \setminus C_n).$$

Q.E.D.

3 Coverings and Compactness

Let (X, d), $A \subseteq X$, and U_α a family of subsets of X. We say that U_α is a *cover* of A if $A \subseteq \bigcup_\alpha U_\alpha$. A *subcover* of the cover U_α is a subfamily of the given family U_α that is also a cover.

Example 11: $X = \mathbb{R}$, $U_n = (-n, n)$ with $U_{n+1} \supset U_n$, so $\bigcup_n U_n = \mathbb{R}$. An *open cover* is one where each U_α is an open subset. A *finite subcover* is a subcover consisting of finitely many sets. Note: $\bigcup_n U_n = \mathbb{R}$ does not have a finite subcover.

Definition 12: Let (X, d), $A \subseteq X$. We say that A is *compact* if for every open cover of A there is a finite subcover.

Why is this useful?

There is a metric version of the Bolzano–Weierstrass theorem for metric spaces.

Theorem 13: *Let (X,d), $A \subseteq X$ with A compact. Then any sequence in A has a convergent subsequence. (Proved next time.)*

Theorem 14: *Let (X,d). If $A \subseteq X$ is compact, then*

1. *A is bounded;*
2. *A is closed.*

Proof of (1). $A \subseteq \bigcup_{x\in A} B_1(x)$. Since A is compact, $A \subseteq B_1(x_1) \cup \dots \cup B_1(x_n)$. If $x \in A$ then there exists $i \in \{1,\dots,n\}$ with $d(x,x_i) < 1$. Set

$$r = \max\{d(x_1,x_i)\} + 1.$$

Want to show that for any $x \in A \implies x \in B_r(x_1)$. Since $x \in B_1(x_i)$ for some i, $d(x,x_i) < 1$, and

$$d(x,x_1) \leq d(x,x_i) + d(x_i,x_1) < 1 + d(x_i,x_1),$$

so $d(x,x_1) \leq r$. **Q.E.D.**

Proof of (2). To prove that A is closed: by the theorem proved earlier, need to show that for $x_n \in A$ with $x_n \to x$, the limit $x \in A$. Suppose not; we want a contradiction.

Fix such an x. Let

$$E = \{y \in X \mid d(x,y) > r\} \text{ open}, \qquad E_{1/n} = \{y \mid d(y,x) > \tfrac{1}{n}\}, \quad n \in \mathbb{N}.$$

Then

$$A \subseteq \bigcup_n E_{1/n} = X \setminus \{x\},$$

since $z \in X \setminus \bigcup_n E_{1/n} \implies d(z,x) \leq \frac{1}{n}$ for all $n \implies d(z,x) = 0 \implies z = x$. Note $E_{1/n} \subseteq E_{1/(n+1)}$.

By compactness, $A \subseteq$ finitely many $\bigcup_n E_{1/n}$, so $A \subseteq E_{1/n_i}$, which means there is no $y \in A$ with $d(y,x) < \frac{1}{n_i}$ for all $y \in A$. Therefore there is no sequence $x_n \to x$ with $x_n \in A$ – a contradiction. **Q.E.D.**

Theorem 15: *In $\mathbb{R}^n$: if A is closed and bounded, then A is compact. (Proved next time.)*

Theorem 16: *Let (X,d). If $A \subseteq X$ is compact and $C \subseteq A \subseteq X$ with C closed, then C is also compact.*

Proof. Since C is closed, $X \setminus C$ is open. Let U_α be a cover of C. Then

$$A \subseteq C \cup (X \setminus C) \subseteq \bigcup_\alpha O_\alpha \cup (X \setminus C).$$

By compactness of A,

$$C \subseteq A \subseteq O_1 \cup \dots \cup O_n \cup (X \setminus C),$$

which implies C must be $\subseteq O_1, \dots, O_n$. **Q.E.D.**