Lecture 12: Convergence in Metric Spaces; Operations on Sets

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1 Course Map

1. $\mathbb{R}$ is an ordered set and field.
2. Sequences and limits.
3. Series.
4. Metric spaces.

2 Metric Spaces (recap)

Definition 1: Let X be a set. A metric is $d : X \times X \to \mathbb{R}$ satisfying, for $x, y, z \in X$:

1. $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Example 2: $X = \mathbb{R}$, $x, y \in \mathbb{R}$, $|x - y| = d(x, y)$.

Example 3: $X = \mathbb{R}^n$, $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ with $x_i, y_i \in \mathbb{R}$:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Let (X, d) be a metric space. A sequence is a map $f : \mathbb{N} \to X$, $x_n = f(n)$. A subsequence of f is given by a strictly increasing map $g : \mathbb{N} \to \mathbb{N}$, written $f(g(n))$.

3 Convergence and Cauchy Sequences

Definition 4: A convergent sequence in a metric space (X, d): if x_n is a sequence in X, then we say $x_n \to x$ if the following holds:

$$\forall \varepsilon > 0 \; \exists N \mid n \geq N, \; d(x_n, x) < \varepsilon.$$

Definition 5: A Cauchy sequence in a metric space (X, d) is a sequence x_n such that

$$\forall \varepsilon > 0\ \exists N \mid m, n \geq N,\ d(x_m, x_n) < \varepsilon.$$

Theorem 6: *In (X, d), any converging sequence is also a Cauchy sequence.*

Proof. Given $\varepsilon > 0$. Since $x_n \to x$, there exists N such that if $n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$. Therefore, if $m, n \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

Cauchy convergence theorem. If $(X, d) = (\mathbb{R}, |\cdot|)$, then we have the converse, i.e. any Cauchy sequence is convergent.

Example 7: $X = (0, 1)$, $d = |\cdot|$. Then $x_n = \frac{1}{n}$ is a Cauchy sequence that is not convergent.

Definition 8 (Cauchy complete)**:** (X, d) is Cauchy complete if every Cauchy sequence is convergent.

4 Metric Balls and Bounded Sets

Definition 9 (Metric ball)**:** Let (X, d), $x \in X$, $r > 0$. Then

$$B_r(x) = \{y \in X \mid d(x, y) < r\}$$

is a metric ball with radius r and center x.

For $X = \mathbb{R}^2$ this is a disk; for $X = \mathbb{R}^3$ a ball (an r-tubular neighborhood, $r > 0$).

Definition 10 (Bounded subset)**:** Let (X, d) and $A \subseteq X$. We say that A is bounded if there exist $x \in X$ and $d > 0$ such that $A \subseteq B_d(x)$.

Lemma 11: *Let (X, d) and let x_n be a Cauchy sequence. Set $A = \{x_n \mid n \in \mathbb{N}\} \subseteq X$. Then A is bounded.*

Proof. Since x_n is a Cauchy sequence (with $\varepsilon = 1$), there exists N such that if $n, m \geq N$, $d(x_n, x_m) < 1$. In particular, for $n \geq N$, $d(x_n, x_N) < 1$. Set

$$r = 1 + \max\{d(x_N, x_i) \mid i = 1, 2, \ldots, N-1\}, \qquad r > 0.$$

Claim $A \subseteq B_r(x_N)$: need to show that for all n, $d(x_n, x_N) < r$. For $n \geq N$ we have $d(x_n, x_N) < 1 \leq r$; for $n \leq N-1$, $d(x_n, x_N) < r$ by construction. **Q.E.D.**

5 Open and Closed Sets

The Bolzano–Weierstrass theorem does not hold in a general metric space, but it does hold if the metric space is compact. We need the notion of an open subset of a metric space.

Definition 12: Let (X, d). O is said to be an open subset if for all $x \in O$ there exists $r > 0$ such that $B_r(x) \subseteq O$.

Note: $\emptyset$ and X are open subsets.

Lemma 13: *Let (X, d), $x \in X$, $r > 0$. Then $B_r(x)$ is open.*

Proof. Let $y \in B_r(x)$, so $d(x, y) < r$. Want to show that for some $s > 0$, $B_s(y) \subseteq B_r(x)$. Set $s = r - d(x, y)$. Take $z \in B_s(y)$; need to show $z \in B_r(x)$. By the triangle inequality,

$$d(x, z) \leq d(x, y) + d(y, z), \qquad d(y, z) < s,$$

so

$$d(x, z) < d(x, y) + s = d(x, y) + (r - d(x, y)) = r,$$

i.e. $d(x, z) < r$. Hence $B_s(y) \subseteq B_r(x)$. **Q.E.D.**

Definition 14: Let (X, d). $C \subseteq X$ is said to be closed if the complement of C is open:

$$C' = \{x \in X \mid x \notin C\}.$$

6 Operations on Subsets of a Given Set

Let X be a set, $U_1, U_2 \subseteq X$.

$$U_1 \cup U_2 = \{x \in X \mid x \in U_1 \text{ or } x \in U_2 \text{ or both}\} \quad \text{(union)},$$

$$U_1 \cap U_2 = \{x \in X \mid x \in U_1 \text{ and } x \in U_2\} \quad \text{(intersection)}.$$

For $U \subseteq X$,

$$X \setminus U = \{x \in X \mid x \notin U\}.$$

Let U_α be a family of subsets. Then

$$\bigcup_\alpha U_\alpha = \{x \in X \mid x \in U_\alpha \text{ for some } \alpha\}, \qquad \bigcap_\alpha U_\alpha = \{x \in X \mid x \in U_\alpha \text{ for all } \alpha\}.$$

Example 15: $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then $\bigcup_n U_n = (-1, 1)$ and $\bigcap_n U_n = \{0\}$.

Lemma 16: *Let X be a set with subsets A, B, A_α. Then:*

1. $X \setminus (X \setminus A) = A$;
2. $X \setminus (\bigcup_\alpha A_\alpha) = \bigcap_\alpha (X \setminus A_\alpha)$;
3. $X \setminus (\bigcap_\alpha A_\alpha) = \bigcup_\alpha (X \setminus A_\alpha)$.

Lemma 17: *$A = B$ is the same as proving $A \subseteq B$ and $B \subseteq A$.*

Lemma 18: *Let (X, d). If O_α are open subsets, then $\bigcup_\alpha O_\alpha$ is also open.*

Proof. Let $x \in \bigcup_\alpha O_\alpha$. Then $x \in O_\beta$ for some β, so there exists $r > 0$ with

$$B_r(x) \subseteq O_\beta \subseteq \bigcup_\alpha O_\alpha.$$

Q.E.D.