Lecture 11: Extreme & Intermediate Value Theorems; Metric Spaces

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1 Continuity (recap)

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. f is said to be continuous if for all $x_0 \in A$ the function is continuous at x_0, i.e. if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \mid |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon, \qquad \delta = \delta(x_0, \varepsilon).$$

2 Extreme and Intermediate Value Theorems

Theorem 1 (Extreme Value Theorem)**:** *$f : [a, b] \to \mathbb{R}$ is continuous. Then there exist $x_M \in [a, b]$ and $x_m \in [a, b]$ such that for all $x \in [a, b]$,*

$$f(x_m) \leq f(x) \leq f(x_M).$$

In English: a bounded function on an interval has a maximum and minimum value.

Theorem 2 (Intermediate Value Theorem)**:** *$f : [a, b] \to \mathbb{R}$ is continuous. Then all values between $f(a)$ and $f(b)$ are achieved.*

Lemma 3: *Let $f : A \to \mathbb{R}$. If f is continuous and x_n is a sequence with $x_n \in A$ and $x_n \to x \in A$, then $f(x_n) \to f(x)$. Equivalently, with $x = \lim_{n\to\infty} x_n$,*

$$f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n).$$

Proof. Want to show $f(x_n) \to f(x)$. First use that f is continuous at x: given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Second, use that $x_n \to x$: for this δ there exists N such that if $n \geq N$, then $|x_n - x| < \delta$. Combining, for $n > N$,

$$|x_n - x| < \delta \implies |f(x_n) - f(x)| < \varepsilon.$$

Q.E.D.

Using the lemma, we will show EVT and IVT.

Proof of EVT. First show that $f([a,b])$ is bounded from above. Suppose not: then for all $n \in \mathbb{N}$ there exists $x \in [a,b]$ with $f(x) \geq n$. Pick such an x_n, so $f(x_n) \geq n$.

Now $x_n \in [a,b]$ is a bounded sequence, so by the Bolzano–Weierstrass theorem it has a convergent subsequence $x_{n_k} \to x$ (where x_{n_k} is a subsequence of x_n). Then $f(x_{n_k}) \to f(x)$, but

$$f(x_{n_k}) \geq n_k \geq n,$$

a contradiction. A similar argument shows f is bounded from below.

Since $f([a,b])$ is a bounded subset, let $M = \sup f([a,b])$. For all $n \in \mathbb{N}$,

$$M - \tfrac{1}{n} < f(x) \quad \text{for some } x \in [a,b],$$

so choosing $x_n \in [a,b]$ with $M - \frac{1}{n} < f(x_n) \leq M$. By the Bolzano–Weierstrass theorem, x_n has a convergent subsequence $x_{n_k} \to x_M$. Then

$$f(x_{n_k}) \to f(x_M) \quad \text{(by continuity of } f\text{)},$$

and $f(x_n) \to M$ (by squeeze), so $f(x_{n_k}) \to M$. By uniqueness of limits, $f(x_M) = M$. **Q.E.D.**

Proof of IVT. Make an assumption: $f(a) < 0 < f(b)$, f continuous. Want to show there exists $x_0 \in [a,b]$ with $f(x_0) = 0$. Let

$$E = \{x \in [a,b] \mid f(y) \leq 0,\ a \leq y \leq x\}.$$

$E \neq \emptyset$ since $a \in E$, and $\sup E \leq b$. Set $x_0 = \sup E$.

Claim 4: $f(x_0) = 0$.

For all $n \in \mathbb{N}$, $x_0 - \frac{1}{n} < x_n \leq x_0$ with $x_n \in E$, so $f(x_n) \leq 0$. Since $x_n \to x_0$,

$$f(x_n) \to f(x_0) \leq 0.$$

Assume $f(x_0) < 0$. Take $\varepsilon > 0$ such that $f(x_0) + \varepsilon < 0$. Then there exists $\delta > 0$ with

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon \implies f(x) < 0.$$

Note: I don't get how this is a contradiction.

But assuming the contradiction holds, $f(x_0) = 0$. **Q.E.D.**

3 Metric Spaces

Definition 5: Let X be a set. A *metric* is a map $d : X \times X \to \mathbb{R}$, where for $x, y \in X$, $d(x,y)$ is the distance between x and y, satisfying:

1. $d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$;
2. $d(x,y) = d(y,x)$ (symmetric);
3. for $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

Example 6: $X = \mathbb{R}$, $x, y \in \mathbb{R}$, $d(x,y) = |x-y|$.

Example 7: $X = \mathbb{R}^2$, $\underline{x} = (x_1, x_2)$, $\underline{y} = (y_1, y_2)$:

$$d(\underline{x}, \underline{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} \quad \text{(Euclidean)}.$$

Example 8 (Box distance)**:** $X = \mathbb{R}^2$, $\underline{x}, \underline{y} \in \mathbb{R}$:

$$d(\underline{x}, \underline{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

Example 9: $X = C([0,1])$. An element in X is a continuous function on $[0,1]$, with

$$d(f,g) = \max_{x\in[0,1]} |f-g|(x).$$

Example 10 (Strange)**:** $X = \mathbb{N}$, $d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$, with $f : \mathbb{N} \to \mathbb{R}$, $f(n) = \frac{1}{n}$.

Example 11 (Wild: French railroad metric)**:** $X = \mathbb{R}^2$, $\underline{x}, \underline{y} \in \mathbb{R}$:

$$d(\underline{x}, \underline{y}) = \begin{cases} |x-y| & \underline{x} = c\underline{y} \text{ or } \underline{y} = c\underline{x} \text{ (same line)}, \\ |x| + |y| & \text{otherwise.} \end{cases}$$

4 Sequences in Metric Spaces

Let (X,d) be a metric space. A map $f : \mathbb{N} \to X$ is a sequence, $f(n) = x_n$. A subsequence is given by a strictly increasing $g : \mathbb{N} \to \mathbb{N}$ with $f(g(k)) = x_{n_k}$.

Let x_n be a sequence in X. We say $x_n \to x$ if

$$\forall \varepsilon > 0 \; \exists N \mid n \geq N, \; d(x_n, x) < \varepsilon.$$

A *Cauchy sequence* in a metric space is the following: for all $\varepsilon > 0$ there exists N such that $n, m \geq N \implies d(x_n, x_m) < \varepsilon$.

Theorem 12: *(X,d) is a metric space. If x_n is a convergent sequence in X, then x_n is also a Cauchy sequence.*

Proof. $x_n \to x$. Given $\varepsilon > 0$, there exists N such that if $n > N$, $d(x_n, x) < \frac{\varepsilon}{2}$. Therefore, if $m, n \geq N$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Q.E.D.

Cauchy completeness. Take $X = (0,1)$, $d(x,y) = |x-y|$, $x, y \in (0,1)$. Then $x_n = \frac{1}{n}$ is a Cauchy sequence, but it does not converge, because $0 \notin X$. So the Cauchy convergence theorem *may* not apply. If it does, the space is said to be *Cauchy complete*.

5 Continuity for General Metric Spaces

Let (X, d) and $F : X \to \mathbb{R}$. F is continuous at $x_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \implies |F(x) - F(x_0)| < \varepsilon.$$

f is said to be continuous if it is continuous at all points.

Example 13: $X = C([0, 1])$, $d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)|$. Let $F : X \to \mathbb{R}$, $F(f) = f(0)$.

Claim 14: *F is continuous.*

$$|F(f) - F(g)| = |f(0) - g(0)| \leq \max_x |f(x) - g(x)| = d(f, g).$$