Lecture 10: Continuous Functions; Exponential Function (cont.)

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1 Power Series (recap)

Let a_n be a sequence. For all x form $\sum_{n=0}^{\infty} a_n x^n$.

Example. $a_n = \frac{1}{n!}$:

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad E \geq 0 \text{ for } x \geq 0, \qquad E(0) = 1, \qquad e = E(1).$$

For $q \in \mathbb{Q}$: $e^0 = 1$, $e^1 = e$, and

$$e^m = \underbrace{e \cdot e \cdots e}_{m \text{ times}}.$$

Assume $E(x)E(y) = E(x+y)$. Then $e^{1/n}$ is defined to be the positive number α with $\alpha^n = e$, and $e^{-m} = \frac{1}{e^m}$. This gives $E(x) = e^x$ for $x \in \mathbb{Q}$.

Pset 5. E is continuous. Moreover, if f, g are two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f|_{\mathbb{Q}} = g|_{\mathbb{Q}}$, then $f = g$ on all of $\mathbb{R}$.

Theorem 1: $E(x+y) = E(x)E(y)$.

Proof in the case where $x, y \geq 0$ (below).

2 Cauchy Product of Series

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ with $a_n, b_n \geq 0$ for all n. Form a third series

$$\sum_{n=0}^{\infty} c_n, \qquad c_n = \sum_{i=0}^{n} a_i\, b_{n-i}.$$

Note that $c_n \geq 0$.

Theorem 2: *If* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are convergent, then* $\sum_{n=0}^{\infty} c_n$ *is also convergent, and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Proof. Let

$$S_n^a = \sum_{i=0}^{n} a_i, \qquad S_n^b = \sum_{i=0}^{n} b_i, \qquad S_n^c = \sum_{i=0}^{n} c_i, \qquad a_i, b_i, c_i \geq 0.$$

Convergence of any of these series is equivalent to S_n^a, S_n^b, S_n^c being bounded (each of the series is increasing, by the monotone convergence theorem).

Expanding the product,

$$\underbrace{(a_0 + a_1 + \cdots + a_n)}_{S_n^a}\underbrace{(b_0 + b_1 + \cdots + b_n)}_{S_n^b}$$

groups by subscript sum:

$$\underbrace{a_0b_0}_{\substack{\text{sum=0}\\ c_0}} + \underbrace{a_0b_1 + a_1b_0}_{\substack{\text{sum=1}\\ c_1}} + \underbrace{a_0b_2 + a_1b_1 + a_2b_0}_{\substack{\text{sum=2}\\ c_2}} + \cdots$$

so

$$(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n) = \underbrace{c_0 + c_1 + \cdots + c_n}_{S_n^c} + \text{(additional terms)}.$$

Thus $S_n^a S_n^b = S_n^c + \beta$ where $\beta \geq 0$, which gives

$$S_n^c \leq S_n^a S_n^b \leq \left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{i=0}^{\infty} b_i\right),$$

using $S_n^a \uparrow \sum_{i=0}^{\infty} a_i$ and $S_n^b \uparrow \sum_{i=0}^{\infty} b_i$. Hence $\sum_{i=0}^{\infty} c_i$ is convergent.

Equality. We have $\sum_{i=0}^{\infty} c_i \leq \left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{i=0}^{\infty} b_i\right)$. Also

$$(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n) = S_n^c + a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots \leq S_{2n}^c,$$

so

$$S_n^c \leq S_n^a S_n^b \leq S_{2n}^c.$$

By the squeeze theorem, $S_n^a S_n^b$ must therefore also converge to $\sum_{i=0}^{\infty} c_i$. **Q.E.D.**

Corollary. If $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ are absolutely convergent, then $\sum_{i=0}^{\infty} c_i$ is also absolutely convergent.

Proof. Absolute convergence: $\sum_{i=0}^{\infty} |a_i|$ and $\sum_{i=0}^{\infty} |b_i|$ are convergent. With $c_i = \sum_{i=0}^{n} a_i b_{n-i}$,

$$|c_i| \leq \left| \sum_{i=0}^{n} a_i b_{n-i} \right| \leq \sum_{i=0}^{\infty} |a_i| \, |b_{n-i}|.$$

Set $\bar{a}_i = |a_i|$, $\bar{b}_i = |b_i|$, $\bar{c}_i = |c_i|$. Then $\bar{a}_i$ is convergent, $\bar{b}_i$ is convergent, so $\bar{c}_i$ must then also be convergent by the theorem. **Q.E.D.**

3 Proof that $E(x+y) = E(x)E(y)$

Theorem 3: $E(x+y) = E(x)E(y)$.

Proof.

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} \frac{y^n}{n!}.$$

Set $c_n = \frac{(x+y)^n}{n!}$, $a_n = \frac{x^n}{n!}$, $b_n = \frac{y^n}{n!}$. By the binomial formula,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}, \qquad \binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!} = \frac{n!}{(n-i)!\,i!}.$$

Then

$$c_n = \frac{(x+y)^n}{n!} = \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \frac{1}{n!} \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} x^i y^{n-i} = \sum_{i=0}^{n} \frac{x^i}{i!} \frac{y^{n-i}}{(n-i)!} = \sum_{i=0}^{n} a_i b_{n-i}.$$

Therefore, by the Cauchy product theorem,

$$E(x+y) = \sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = E(x)E(y).$$ **Q.E.D.**

4 Continuity

Let $f : A \to \mathbb{R}$, $x_0 \in A$. f is continuous at x_0 if the following holds: for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Example. A function defined on all of $\mathbb{R}$ that is nowhere continuous:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For $x_0 \in \mathbb{Q}$: since $\sqrt{2}$ is not rational, the points $x_0 + \frac{\sqrt{2}}{n}$ (for $n \geq N$) are irrational and approach x_0. If $\varepsilon < 1$, then for any δ we can choose $x \in \mathbb{R} \setminus \mathbb{Q}$ with $|x - x_0| < \delta$, giving

$$|f(x) - f(x_0)| = |f(x)| = 1.$$

This proves that f is discontinuous at all rational points.

If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then $f(x_0) = 1$, and for all $\delta > 0$ there exists $x \in \mathbb{Q}$ with $|x - x_0| < \delta$, giving

$$|f(x) - f(x_0)| = |0 - 1| = 1.$$

Therefore f is also discontinuous at all irrational points.

5 Algebraic Properties (proofs)

If f, g are continuous functions and c is a constant:

1. $f + g$ is also continuous.
2. cf is continuous.
3. fg is continuous.
4. $f \neq 0 \implies \dfrac{1}{f}$ is continuous.
5. $f : A \to B$, $g : B \to \mathbb{R}$: $h(x) = g(f(x))$ is also continuous.

Proof of (5). Given $\varepsilon > 0$, want to show h is continuous at x_0. Let $y_0 = f(x_0)$. Since g is continuous at y_0, there exists $\delta_g > 0$ such that

$$|y - y_0| < \delta_g \implies |g(y) - g(y_0)| < \varepsilon.$$

From $\delta_g > 0$ and the continuity of f, there exists $\delta_f > 0$ such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \delta_g,$$

where $|f(x) - f(x_0)| = |f(x) - y_0|$. Hence

$$|g(f(x)) - g(y_0)| < \varepsilon.$$

Since $g(f(x)) = h(x)$ and $g(y_0) = g(f(x_0)) = h(x_0)$,

$$|x - x_0| < \delta_f \implies |h(x) - h(x_0)| < \varepsilon.$$

Q.E.D.

Proof of (4). $f \neq 0$, $x_0 \in A$, f continuous at x_0. Want to show $\dfrac{1}{f}$ is continuous at x_0:

$$\left| \frac{1}{f(x)} - \frac{1}{f(x_0)} \right| = \left| \frac{f(x_0) - f(x)}{f(x_0) f(x)} \right|.$$

Since $f(x_0) \neq 0$, there exists $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \implies |f(x) - f(x_0)| < \frac{|f(x_0)|}{2}.$$

Writing $f(x_0) = f(x_0) - f(x) + f(x)$,

$$|f(x_0)| \leq |f(x_0) - f(x)| + |f(x)|,$$

so

$$|f(x)| \geq |f(x_0)| - |f(x_0) - f(x)| \geq \tfrac{1}{2}|f(x_0)|.$$

Q.E.D.