Lecture 1: Introduction to Real Numbers

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1 Introduction

The goal of this course is to learn how to write a proof and prove theorems with the real numbers. This lecture is about the real numbers

2 Real Numbered Sets

The natural numbers are denoted by $\mathbb{N}$. This is the set of all positive integers. (1, 2, 3, 4, etc.) The integers are denoted by $\mathbb{Z}$. This is the set of all integers, including 0. The rational numbers are denoted by $\mathbb{Q}$. This is the set of all rational numbers, or numbers that can be written as a fraction. The set of all real numbers is defined by $\mathbb{R}$. This is what the entire course is about.

3 Properties of the Rational Numbers

Properties of the rational numbers include the following.

3.1 Representation

$$m_1, m_2 \in \mathbb{Z}.$$

$$n_1, n_2 \in \mathbb{N}.$$

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \Leftrightarrow m_1 n_2 = m_2 n_1.$$

$$\frac{m_1}{n_1} < \frac{m_2}{n_2} \Leftrightarrow m_1 n_2 < m_2 n_1.$$

$$\frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2}.$$

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1 m_2}{n_1 n_2}.$$

3.2 Operations

$$p, q \in \mathbb{Q}.$$
$$p + q \in \mathbb{Q}.$$
$$p \cdot q \in \mathbb{Q}.$$

We must now check to see if these operations are valid, by checking to see if they function independent of the representation of the numbers. This is because these operations that we defined should work the same, regardless of what we put as inputs.

Let us start by defining two different statements that we postulate should be equal.

$$\frac{m_1}{n_1} \cdot \frac{p_1}{q_1}. \tag{1}$$

$$\frac{m_2}{n_2} \cdot \frac{p_2}{q_2}. \tag{2}$$

Remember, for this to be true, these statements must hold

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \Leftrightarrow n_1 m_2 = n_2 m_1 \tag{3}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \Leftrightarrow p_1 q_2 = p_2 q_1. \tag{4}$$

This is because of the rules of representation from the above section. We can rewrite 1 and 2 together as

$$\frac{m_1 p_1}{n_1 q_1} = \frac{m_2 p_2}{n_2 q_2}. \tag{5}$$

This is the statement we want to prove. We know from 3 and 4 that this is true if and only if

$$m_1 p_1 n_2 q_2 = m_2 p_2 n_1 q_1. \tag{6}$$

By the method of comparing coefficients, we get

$$m_1 n_2 = m_2 n_1, p_2 q_1 = q_2 p_1 \tag{7}$$

These are the conditions which satisfy our postulate.

Q.E.D.

We can prove that addition works this way too, but it is a little messier, and so that will be left as an exercise to the reader. :D

4 Fields

4.1 Definition

A field, $\mathbb{F}$, is a set with two operations. For our purposes, we suggestively denote these operations as "+" and "$\cdot$". These are left purposefully vague for now, and if you are wondering why that is, try to come up with some other, more precise definition yourself. You will find that this rather vague definition is actually the most precise we can get without convoluding things too much or contradicting ourselves.

We will also suggestively label the "+" operation as addition and the "$\cdot$" operation as multiplication. In a field, these operations each have 5 properties. For the following statements $x, y, z \in \mathbb{F}$.

Properties of addition:

1. $x + y \in \mathbb{F}$. (Operative property)
2. $x + y = y + x$. (Abelian property)
3. $(x + y) + z = x + (y + z)$. (Associative property)
4. $\exists$ "0" $\mid x + 0 = x$. (Neutral property)
5. $\forall\ x,\ \exists\ (-x) \mid x + (-x) = 0$. (Inverse property)

Properties of multiplication:

1. $x \cdot y \in \mathbb{F}$. (Operative property)
2. $x \cdot y = y \cdot x$. (Abelian property)
3. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. (Associative property)
4. $\exists$ "1" $\mid x \cdot 1 = x$. (Neutral property)
5. $\forall\ x \in \mathbb{F}\ \sim \{0\}\ \exists\ \frac{1}{x} \mid x \cdot \frac{1}{x} = 1$. (Inverse property)

You might notice that each of their properties are similar to each other, but not exactly the same. This may be helpful when needing to memorize these properties. However, the properties don't interact with each other. We need to define a way to get these properties to join, so that they don't exist in two different "worlds." This brings rise to an eleventh property. The distributive property.

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

You might also notice that a lot of these properties you have already heard of (the abelian property is also known as the communicative property). These properties are taught in grade school. But now that we are creating a more rigorous understanding of our real number system, we must also create a more rigorous definition of each of these properties.

4.2 Exploration into the Consequences of Fields

Let us use these definitions to prove that there exists only one zero element in every field.

Prove that $\forall\ \mathbb{F},\ \exists$ *a single zero element.*

$$\text{Assume } 0_1, 0_2 \text{ are both zero elements.}$$

$$0_1 + x = x \Rightarrow 0_1 + 0_2 = 0_2.$$

$$0_2 + x = x \Rightarrow 0_1 + 0_2 = 0_1.$$

$$\therefore 0_1 = 0_2.$$

Q.E.D.

To help create an example of what a field is, the rational numbers are a field. The natural numbers and integers are <u>not</u> a field. This is because they each lack an inverse element.

$$\mathbb{Q} \text{ is a } \mathbb{F}.$$

$$\mathbb{N}, \mathbb{Z} \text{ are not a } \mathbb{F}.$$

5 Ordering

5.1 Definition

An ordered set is a set that has some operation, suggestively labelled "$<$", that demonstrates some kind of correct "order." For example, if $x, y \in \mathbb{S}$ then one of the following holds:

1. $x = y$.
2. $x < y$.
3. $y < x$.

Another important property to know is the transitive property, which is stated below.

We define $\mathbb{S}$ to be an ordered set.

$$x, y, z \in \mathbb{S}.$$

$$x < y,\ y < z \Rightarrow x < z.$$

To create the definition of an ordered field, we just extend this idea to our understanding of a field. It now has a twelfth property that comes with the "<" operation. This ordered property then leads us to two conclusions.

1. $x < y \Rightarrow \forall\ z \in \mathbb{F}, x + z < y + z.$
2. $0 < x, y \Rightarrow 0 < xy.$

5.2 Exploration into the Consequences of Ordering

Let us now prove a statement.

Prove that if $\mathbb{F}$ is an ordered field, and $x, y, z \in \mathbb{F}$, and $x < y$, then $x \cdot z < y \cdot z$.

Let us start by assuming the proposition $x \cdot z < y \cdot z$.

This is equivalent to $x \cdot z + (-x \cdot z) < y \cdot z + (-x \cdot z)$.

$$0 < z \cdot (y + (-x)).$$

$$0 < y + (-x) \Leftrightarrow x < y.$$

Q.E.D.

We did all this work to lead up to this one statement. $\mathbb{R}$ is a complete, ordered field that contains the rational numbers, $\mathbb{Q}$. It is with this that lecture 1 ended.